ICFP - Université Pierre et Marie Curie

Lecture notes. Year 2016-2017

Cosmology and structure formation

Author:
Benoit Semelin

December 8, 2016

Contents

1 Introduction **7**

1.1 Olbers' paradox . . . 7

1.2 Hubble's law . . . 7

1.3 Discovery of the cosmic microwave background . . . 9

2 The expanding universe: FLRW formalism **11**

2.1 Basics of General Relativity for cosmology . . . 11

2.1.1 The metric tensor . . . 11

2.1.2 Einstein's equation . . . 12

2.2 The cosmological principle and the Friedmann-Lemaître-Roberston-Walker metric . . . 13

2.2.1 Cosmological principle . . . 13

2.2.2 The Friedmann-Lemaître-Robertson-Walker metric . . . 14

2.2.3 Meaning of the expansion factor . . . 15

2.2.4 Momentum decay in a flat FLRW universe . . . 17

2.3 The Einstein equation in the FLRW metric . . . 17

2.3.1 Computation of the Einstein tensor . . . 17

2.3.2 The stress-energy tensor in cosmology . . . 18

2.4 The Friedmann equations . . . 20

2.4.1 Proper (physical) and comoving distances . . . 20

2.4.2 The dust universe . . . 21

2.4.3 The Einstein - de Sitter universe . . . 21

2.4.4 The case of curved space-time . . . 22

2.4.5 The cosmological constant and dark energy . . . 23

2.4.6 The radiation universe . . . 23

2.5 The standard ΛCDM model of the universe . . . 25

3 The cosmic microwave background (CMB) **27**

3.1 Black-body radiation in a expanding universe . . . 27

4 Inflation **31**

4.1 Evolution of the Universe curvature (the flatness coincidence) . . . 31

4.2 The CMB isotropy paradox . . . 32

4.3 A scalar field as the source of inflation . . . 33

4.3.1 Stress-energy tensor of a scalar field . . . 33

4.3.2 Evolution of the scalar field: the Klein-Gordon equation . . . 34

4.3.3 Conditions for inflation to occur . . . 34

4.3.4 A simple case of self-interacting scalar field 35

5 Using supernovae as cosmic candles **37**
5.1 Standard candles 37
5.2 The brightness-distance relation 38

6 Writing classical physics equations in a flat, homogeneous, expanding metric **41**
6.1 Regime of validity 41
6.2 Rewriting physical quantities and the differential operators in comoving space 41
6.3 Continuity equation 42
6.4 Poisson's equation 43
6.5 Euler's Equation 43

7 The growth of density perturbations **45**
7.1 Newtonian perturbation theory 45
7.2 Solutions 46
7.3 The initial density perturbation field 47
7.3.1 Gaussian random fields 47
7.3.2 2-point correlation function and power spectrum 48
7.3.3 Linear evolution of the initial density fluctuations power spectrum 49

Forewords

1 Introduction

Cosmology is the study of the evolution of the Universe on large scales. Here, large means on scales much larger than the typical distances between galaxies, that is tens of Mpc. But why would such a science even be needed? When did it become clear that observations do not agree with an infinite, homogeneous and stationary Universe? Both from the observational and the theoretical point of view, one can say that cosmology was born in the beginning of the 20$^{\text{th}}$ century. But interestingly, unanswered questions (whose answers lie in modern cosmology) appeared much earlier than that.

1.1 Olbers' paradox

Olbers' paradox can be stated quite simply: "Why is the night sky dark?". The question is actually much older than Olbers formulation (1823). Kepler (for example) was wondering the same thing as early as 1610. The paradox arises when considering a homogeneous stationary and infinite universe. Then the content of the universe can be described with a constant number density n of sources of light of luminosity L. The flux received by an observer from a source of light located at a distance r is then $\frac{L}{4\pi r^2}$. Summing over all sources, the total flux is:

$$F = \int_0^\infty \frac{n\,L}{4\pi r^2} 4\pi r^2 dr = +\infty. \tag{1.1}$$

Allowing for the fact that stars are not point-like and can screen other stars further along the line of sight, we get a night sky that should be as bright as the typical surface of a star. Neither Kepler nor Olbers gave a satisfactory answer to the paradox. Kepler assumed that there were no stars beyond a certain distance, which conflicts with homogeneity but was reasonable at the time, and Olbers assumed that light was absorbed along the way by interstellar gas, which would induce an increase in the temperature of the gas and break the assumption that the Universe is stationary. The point is that Olbers paradox cannot be resolved in a homogeneous, stationary and infinite universe. Modern cosmology solves it by getting rid of stationary Universe assumption. Let's also mention that until only 20 years ago another solution was to postulate a universe with a fractal content (thus getting rid of homogeneity without introducing a center). While this is compatible with observations up to scales of a few tens of Mpc, recent galaxy surveys show that homogeneity is reached on > 100 Mpc scales.

1.2 Hubble's law

In 1929, Edwin Hubble found a correlation between the distance of 24 nearby galaxies and their radial velocities. The distances were estimated using Cepheid stars when the image resolution was good enough that individual star could be identified. Indeed, Cepheid stars have an average intrinsic luminosity that can be measured from the period of its fluctuation. Measuring the received flux then

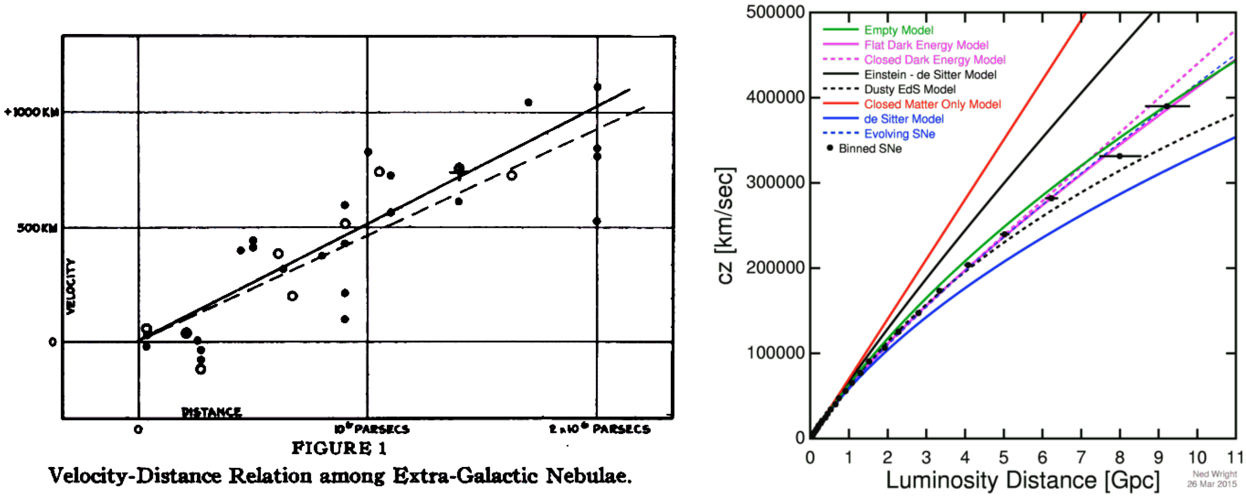

Figure 1.1: On the left, the original velocity-distance diagram by Hubble (1929). On the right, a modern version (credit: Ned Wright, compiled with data from Betoule et al. 2014) of the diagram, The black dots are a binned representation of a sample of 740 supernovae of type Ia.

yields the distance (assuming negligible absorption along the line of sight). In other cases, much more approximate estimators of the distance were used. Although no reference is given in the original paper, the radial velocities were most likely obtained through measurements of the Doppler shift of spectral lines by V. M. Slipher. Hubble's insight was to recognize that the receding velocities (corrected from the Sun's own velocity) were proportional to the distance of the galaxies:

$$\boxed{v = H_0 d} \tag{1.2}$$

H_0 is called the Hubble constant. Hubble's original estimation of H_0 was wildly wrong (more than 7 times the current value), mainly because of the strongly underestimated distances of the most distant galaxies in his sample. Nevertheless, this was the first observational hint that the universe as a whole is expanding homogeneously. Modern cosmology still probes this relation, using different but equivalent quantities, applied to the case type Ia supernovae whose distances can be accurately computed (they have a well known peak luminosity). The original and modern versions of the velocity-distance relation are shown in fig. 1.2.

The notion of redshift

In cosmology the redshift of an object is a quantity that is used very often. It is originally an observational measurement, quantifying how much the observed wavelength of an emission line from a distant galaxy is shifted towards the red side of its rest-frame value.

$$\boxed{z = \frac{\lambda_{\mathrm{obs}} - \lambda_{\mathrm{rest}}}{\lambda_{\mathrm{rest}}}} \tag{1.3}$$

It is tempting to interpret it as a simple Doppler effect ($z = \frac{v}{c}$ where v is the receding velocity of the observed galaxy), as in Hubble's analysis. However, redshifts larger than 1 are observed (not by Hubble...). While this is still possible using $v \sim c$ and the appropriate special relativistic formula, General Relativity teaches us that the main part of the receding velocity, called the Hubble flow, is not an actual velocity but the result of the expansion of the universe.

Since there is a one-to-one relation between the redshift and the distance of an object through Hubble's law (and thus a one-to-one relation to the time when the light we observe was emitted by the object) redshift has actually come to replace time has a variable for tracking the evolution of the universe.

1.3 Discovery of the cosmic microwave background

Hubble's observation drew a picture of a universe in expansion. Even without resorting to a self-consistent theoretical model (provided by general relativity), consequences can be inferred. If we extrapolate the expanding behaviour in time, the Universe was much denser in the past. Since a large fraction of the baryonic content of the universe is hydrogen gas and that, by definition, the universe does not exchange heat with anything, the gas undergoes an adiabatic expansion. Thus it was denser AND hotter in the past. Then, sufficiently early on, it must have been in the form of a plasma. Photons interact much more strongly with charged particles (protons and electrons) than with neutral atoms. At the time, matter and radiation must have been in thermal equilibrium. When matter cooled down sufficiently for hydrogen to recombine (at a few thousands K depending on the density) the matter-radiation interaction became much weaker and the mean free path of photons very large. Thus we should be able to observe those relic photons that last interacted with matter at the epoch of recombination. At the time, they had a black-body spectrum (arising from the thermodynamical equilibrium with matter). It can be shown that the effect of expansion preserves the black-body spectrum while inducing a decrease in the effective temperature, shifting the peak wavelength into the microwave range.

There were theoretical predictions as early as the 50's that this thermal radiation bath, named the Cosmic Microwave Background (CMB), existed. In 1964 Wilkinson and Roll, colleagues of R. Dicke at Princeton university, started building a radiometer to measure the expected signal. At the same time Penzias and Wilson, working at Bell Labs, had built a very sensitive antenna/receiver combination for radio-astronomy observations. These last two measured an isotropic "noise" they could not interpret as instrumental or of terrestrial origin. Their measurement was performed at $\lambda = 7.5$ cm and was consistent with a black-body temperature of 3.5 ± 1K. They contacted R. Dicke who suggested that it could be the expected CMB. Two years later, Roll and Wilkinson confirmed the interpretation by measuring an isotropic emission at 3.2 cm consistent with a 3.0 ± 0.5K black-body. Since this original observations a number of dedicated instruments have been built to observe the CMB in ever greater details, as it is a fossil of the early universe where a wealth of information about cosmology is encoded. The most famous are probably the three satellite programs COBE (1990), WMAP (2001) and Planck (2009), that were able to observe the CMB near its peak emission ($\sim$ 1mm), where the atmosphere has strong absorption bands. The near-perfect fit to a black-body spectrum as measured by COBE and the tiny temperature anisotropies of the CMB measured by Planck can be seen on fig. 1.3.

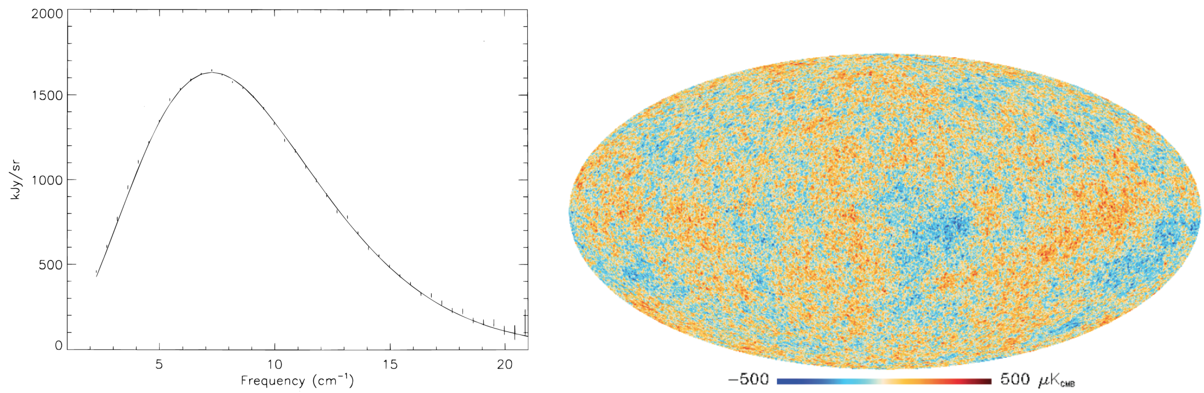

Figure 1.2: Left panel: fit of the COBE data to a 2.73 K blackbody spectrum (Fixsen et al., 1996, AJ, 473, 576). Right panel: temperature fluctuations map of the sky observed by Planck (Planck collaboration, 2014, A&A, 571, A1).

2

The expanding universe: FLRW formalism

2.1 Basics of General Relativity for cosmology

We will here summarize the fundamental aspects of the theory of General Relativity (hereafter GR). Indeed, modern cosmology derives from the simplest application of GR: the case with the highest possible symmetry level. This section is a meant as a refresher and some prior knowledge of special and general relativity is assumed.

2.1.1 The metric tensor

Special relativity was build to account for a striking observational fact: the speed of light measured in all inertial frames is the same. In other words, the relation $c^2dt^2 - dx_1^2 - dx_2^2 - dx_3^2 = 0$ should hold when moving from one inertial frame to another (the x_i designate Cartesian coordinates, t is the time and c the speed of light). As we know, this invariance requirement is the core of the Lorentz transformation. But on an even more fundamental level, it states that a 4-dimensional space-time where differential distance is defined as $ds^2 = c^2dt^2 - dx_1^2 - dx_2^2 - dx_3^2$ is the natural framework in which the Lorentz transformation operates. If we denote the position 4-vector characterizing an event $d\mathbf{x} = (ct, x_1, x_2, x_3)$, we can rewrite the differential (proper) distance between two events:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{2.1}$$

where η is the Minkowsky metric tensor:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.2}$$

Note the choice of signature $(1, -1, -1, -1)$, common in the field of cosmology.

Within special relativity, the metric is unique. it is uniform over the full 4D spacetime (the components of η are constants). General relativity relaxes this condition. For every event of space-time, there exists a reference frame, the frame of a free-falling observer, whose metric tensor is locally of the Minkowsky form. That a free-falling observer can locally work with special relativity is actually a formulation of the equivalence principle and thus of postulate of GR. Moving away macroscopically from the event in time or space, special relativity fails and the form of the metric tensor changes, its coefficients becoming functions of space and time. Global inertial frames do not have meaning in GR. In the general case, the metric tensor is denoted g in GR, and the differential proper distance between two events reads:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.3}$$

with $g_{\mu\nu}$ a function of time and space coordinates.

2.1.2 Einstein's equation

How the metric changes when moving away from a free falling observer is quantified by Einstein's equation, the second main postulate of GR. It states that this change is determined by the local matter and energy content of the universe, or in other words the stress-energy (or energy-momentum) tensor.

The stress-energy (energy-momentum) tensor

One of the most useful predictions of Special Relativity is the conservation of the norm of the 4-momentum of an isolated system. The 4-momentum of a point particle can be written $\mathbf{p} = (E/c, p_1, p_2, p_3)$, where E is the energy, and the $p_i = \gamma m v_i$ are the relativistic 3-momentum components, with γ the Lorentz factor, m the mass and v_i the components of the 3-velocity. Then the norm of the 4-momentum is $\eta_{\mu\nu} p^\mu p^\nu = \frac{E^2}{c^2} - p^2 = m^2 c^2$, where p^2 stands for the squared norm of the 3-momentum.

For a continuous medium such as a fluid (a relevant approximation in cosmology) the interesting quantity is the 4-vector field of density of 4-momentum. This quantity also obeys a conservation law. Using an Eulerian description of the fluid, this law will take the form of conservation differential equations. To express the conservation of any continuous quantity it is necessary to introduce the corresponding fluxes. In our case: the fluxes of momentum. This is exactly the information provided by the stress-energy tensor.

$$T = \begin{pmatrix} T^{00} = \text{energy density} & \begin{array}{l} cT^{01} = \text{energy flux} \\ \text{through } x_1 = \text{cst} \end{array} & \begin{array}{l} cT^{02} = \text{energy flux} \\ \text{through } x_2 = \text{cst} \end{array} & \begin{array}{l} cT^{03} = \text{energy flux} \\ \text{through } x_3 = \text{cst} \end{array} \\ T^{10}c^{-1} = \text{density of } p_1 & \begin{array}{l} T^{11} = \text{flux of } p_1 \\ \text{through } x_1 = \text{cst} \end{array} & \begin{array}{l} T^{12} = \text{flux of } p_1 \\ \text{through } x_2 = \text{cst} \end{array} & \begin{array}{l} T^{13} = \text{flux of } p_1 \\ \text{through } x_3 = \text{cst} \end{array} \\ T^{20}c^{-1} = \text{density of } p_2 & \begin{array}{l} T^{21} = \text{flux of } p_2 \\ \text{through } x_1 = \text{cst} \end{array} & \begin{array}{l} T^{22} = \text{flux of } p_2 \\ \text{through } x_2 = \text{cst} \end{array} & \begin{array}{l} T^{31} = \text{flux of } p_2 \\ \text{through } x_3 = \text{cst} \end{array} \\ T^{30}c^{-1} = \text{density of } p_3 & \begin{array}{l} T^{31} = \text{flux of } p_3 \\ \text{through } x_1 = \text{cst} \end{array} & \begin{array}{l} T^{32} = \text{flux of } p_3 \\ \text{through } x_2 = \text{cst} \end{array} & \begin{array}{l} T^{33} = \text{flux of } p_3 \\ \text{through } x_3 = \text{cst} \end{array} \end{pmatrix} \tag{2.4}$$

Looking, for example, at the component of the first line in the stress-energy tensor, it is indeed possible to relate them through a conservation law. The rate of change of the energy content of an infinitesimal cubic volume of size ϵ is:

$$\epsilon^3 \frac{\partial T^{00}}{\partial t}$$

Per unit time, the energy inflow and outflow through the 6 faces of the cubic volume are:

$$\epsilon^2 c T^{01}(x), \quad -\epsilon^2 c T^{01}(x+\epsilon), \quad \epsilon^2 c T^{02}(y), \quad -\epsilon^2 c T^{02}(y+\epsilon), \quad \epsilon^2 c T^{03}(z), \quad \text{and} - \epsilon^2 c T^{03}(z+\epsilon)$$

or in total,

$$-\epsilon^3 c \left(\frac{\partial T^{01}}{\partial x} + \frac{\partial T^{02}}{\partial y} + \frac{\partial T^{03}}{\partial z} \right).$$

Equating this to the rate of change of energy content yields:

$$\frac{\partial T^{00}}{\partial t} + c \frac{\partial T^{0i}}{\partial x_i} = 0.$$

Or, using the summation over the 4 space-time components implied by the greek indexes:

$$\frac{\partial T^{0\alpha}}{\partial x_\alpha} = 0$$

The same procedure can be applied to the conservation of momentum, yielding the compact, general conservation law:

$$\boxed{\frac{\partial T^{\beta\alpha}}{\partial x_\alpha} = 0} \tag{2.5}$$

When applied to the stress-energy tensor of a perfect fluid this equation is equivalent, in the framework of special relativity, to both the matter conservation equation and Euler's equation. To write this tensor equation in the framework of GR, we simply replace the derivation operator by the covariant derivative:

$$\boxed{\boldsymbol{\frac{\partial T^{\beta\alpha}}{\partial x_\alpha} + \Gamma^{\alpha}_{\ \mu\alpha} T^{\beta\mu} + \Gamma^{\beta}_{\ \mu\alpha} T^{\mu\alpha} = T^{\beta\alpha}_{\ \ ;\alpha} = 0}} \tag{2.6}$$

where the Γ are the Christoffel symbols or metric connections. We refer the reader to a full course in GR for complete definitions of the metric connections and covariant derivative. We will simply say that the covariant derivative computes the difference between from the observed change in a quantity (vector or tensor) and the change due to a (parallel) transport an infinitesimal path in a non-flat geometry, and thus recovers the intrinsic change of the quantity. In other words, it disentangles the change due to gravitation from those due to other processes.

Einstein equation

The form of the stress-energy tensor varies depending on the assumed properties of content of space-time (dark matter, gas, radiation, etc...) but the conservation equation 2.6 always remains valid. Consequently if a second order tensor describing the geometry of space-time is proportional to the stress energy tensor, as a postulate of GR, it should obey the same equation. This is not the case of the Ricci tensor, the unique contraction of the 4th order Riemann tensor describing the curvature of space-time. This is why Einstein built the so-called Einstein tensor by subtracting its divergence from the Ricci tensor and stating that it should be proportional to the stress energy tensor. Once again we refer the reader to a full course in GR for the exact definitions of Riemann, Ricci and Einstein tensors. We will denote the Einstein Tensor G. The proportionality constant between he Einstein tensor and the stress energy tensor, beyond dimensionality requirements, is chosen such that Newton's Law can be recovered in the weak gravitational field limit. Finally Einstein's equation can be written:

$$\boxed{\boldsymbol{G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}}} \tag{2.7}$$

2.2 The cosmological principle and the Friedmann-Lemaître-Roberston-Walker metric

2.2.1 Cosmological principle

The cosmological principle is a postulate upon which standard modern cosmology is built. It states that:

On sufficiently large scales, the universe is spatially homogeneous and isotropic.

Let us first mention that this principle is still being challenged. Until recently, for example, the possibility of the distribution of matter in the universe having a fractal structure (that is showing pattern on *any* scale) was considered. However, as observations of the distribution of galaxies farther and farther away from us improved, it appeared that indeed, on scales larger than 50 Mpc, fluctuations in the distribution of matter (as inferred from observed galaxies) decreased steadily with increasing scale. For example, if one measures the average density in a sphere of radius R and evaluates the variance of the fluctuations of this quantity when the center of the sphere is chosen at random, it will appear that the variance decreases towards zero with increasing R.

Ignoring completely what happens at "insufficiently" large scales is the first step to build a cosmological solution to GR. We will impose homogeneity and isotropy to the metric and the distribution of matter (that is the stress-energy tensor).

Before we proceed to present this cosmological solution, it is interesting to ponder why the cosmological principle should apply at all. While gravity, the dominant force shaping the large scale structures of the universe, tends to destroy homogeneity (a slightly overdense region in an homogeneous universe will attract matter and grow more overdense) it takes longer and longer to achieve this as one looks at larger scales. As large scales look more homogeneous in observations, one can surmise that the universe was much more homogeneous in the past on all scales and is growing less so under the action of gravity. But why was it homogeneous in the first place? A logical answer is that some interaction other than gravity helped (like, for example, pressure in a perfect gas). However, GR states that no interaction can propagate faster than the speed of light. We will see that in a theory where the evolution of the universe arose from an initial Big Bang, this creates a potential difficulty: homogeneization should be possible only up to scales equal to the distance travelled by a photon between the Big Bang and the present.

2.2.2 The Friedmann-Lemaître-Robertson-Walker metric

In applying the cosmological principle, we will rephrase it so: *at any given instant in time, all experiments and observations will produce the same results anywhere in the universe.* The large scale restriction in now implied. We will now derive the implications for the form of the metric tensor. Let's consider two observers with fixed spatial coordinates at a distance dl from each other. At time t they send each other a light pulse, and mesure the proper time until they receive the pulse from the other. Since they are stay at fixed coordinates, this proper time will be $d\tau = \sqrt{g_{00}}dt = \sqrt{g_{00}}\, dl/c$, and it should be the same for both. Thus g_{00} should not depend on the spatial coordinates.

A careful mathematical definition of a spatially homogeneous and isotropic spacetime shows that is can be foliated with a one parameter family of space-like homogeneous and isotropic surfaces. The world line of any observer that can verify isotropy is *perpendicular* to the space-like surfaces. If not, the intersection of the space-like surface and the surface perpendicular to the world line of the isotropic observer creates a special direction that breaks isotropy. Thus the metric can be written in the form:

$$ds^2 = g_{00}(t)c^2dt^2 - dl^2 \tag{2.8}$$

where dl is the line element for the 3D space-like surface. Let us now find an explicit form for dl^2.

In a 3D non-Euclidean space that is homogeneous and isotropic the curvature of any geodesic, at any point along the geodesic, should be the same. Let's denote R as the associated radius of curvature. If we embed this non-Euclidean 3-space in a Euclidean 4-space with Euclidean coordinates (x,y,z,w), the non-Euclidean 3-space of constant positive Gaussian curvature R^{-3} (actually an hypersphere of radius R) can be described by the equation:

$$x^2 + y^2 + z^2 + w^2 = R^2 \tag{2.9}$$

Or, using the spherical coordinate (r, θ, ϕ) such that $r^2 = x^2 + y^2 + z^2$:

$$r^2 + w^2 = R^2 \tag{2.10}$$

The infinitesimal distance between two points in the euclidean 4-space can be written:

$$dl^2 = dx^2 + dy^2 + dz^2 + dw^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2) + dw^2 \tag{2.11}$$

If these two points are constrained to be on our non-Euclidean 3-space, differentiating eq. 2.10 gives the relation $rdr + wdw = 0$, then $r^2dr^2 = w^2dw^2$. Combining again with eq. 2.10, one gets:

$$dw^2 = \frac{r^2dr^2}{R^2 - r^2} \tag{2.12}$$

Then the infinitesimal distance between 2 points on the 3D non-Euclidean space of constant positive curvature R^{-3} is:

$$dl^2 = \frac{R^2}{R^2 - r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2) \tag{2.13}$$

Finally, noting that the radius of curvature can actually be a function of time and introducing the rescaled variable $\sigma = r/R(t)$, we get the expression for the Friedmann-Lemaître metric:

$$ds^2 = g_{00}(t)c^2dt^2 - R^2(t)\left(\frac{1}{1-\sigma^2}d\sigma^2 + \sigma^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right) \tag{2.14}$$

It is then possible to set g_{00} to 1 by redefining the time variable. The above relation was established in the case where space-like sections of space-time have uniform positive curvature. The case where the curvature is zero simply yields:

$$ds^2 = c^2dt^2 - R^2(t)(dx^2 + dy^2 + dz^2) = c^2dt^2 - R^2(t)(d\sigma^2 + \sigma^2(d\theta^2 + \sin^2\theta\ d\phi^2)). \tag{2.15}$$

The case with constant negative curvature is more tricky. Indeed such a manifold cannot be globally embedded in a Euclidean 4-space. One has to embed it in a Minskowsky-like 4-space, with metric relation $dl^2 = dx^2 + dy^2 + dz^3 - dw^2$ where it is defined by the relation $x^2 + y^2 + z^2 - w^2 = -R^2$. A metric relation valid for all three cases can be written:

$$\boxed{\boldsymbol{ds^2 = c^2dt^2 - R^2(t)\left(\frac{1}{1-K\sigma^2}d\sigma^2 + \sigma^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right),}} \tag{2.16}$$

where $K = -1, 0, +1$ corresponds to the negative, zero and positive curvature cases. This metric is known as the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. It is the basic framework for the standard cosmological model. While manageable in terms of analytical development, it is still more complicated than the simple expanding-contracting flat universe case. Since a zero curvature is still compatible with the ever stronger constraints from cosmological observations we will from now on focus our analytical developments on the flat universe (using Cartesian rather than spherical coordinates), mentioning results for the general case when necessary.

2.2.3 Meaning of the expansion factor

The $R(t)$ notation is natural when we consider a hyperspheric universe, but not especially so for a flat universe. To conform with usual modern cosmology notation, we will replace it by $a(t)$, called the expansion factor. Then the metric tensor is:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix} \tag{2.17}$$

It is important to grasp the physical meaning of the expansion factor. First, obviously from eq. 2.16, the expansion relates the proper (physical) distance and the coordinate (also called comoving) distance through $d\mathbf{s} = a(t)d\mathbf{x}$. Two particles with fixed spatial coordinates will see their physical distance increase (or decrease) with time. **By convention the expansion factor is equal to one at present time.**

All definitions of physical quantities (e.g. density) and local laws governing physical process (e.g. Euler's equation, laws of thermodynamics, etc.) should be written in terms of proper (physical) distances. For example if a masse m occupies a comoving volume Δx^3 the physical density is $\frac{m}{a^3 \Delta x^3}$.

Let us consider an observer emitting a monochromatic light signal at wavelength λ_0 between times t_0 and $t_0 + \delta t_0$. A second observer, located at a comoving distance ΔX receives the signal between t_1 and δt_1. Relating the physical distance of the observers to t_0 and t_1 is inconvenient since it changes in time: the distance already covered by photons on their way keeps expanding! It is much simpler to related ΔX to t_0 and t_1. Remembering that c is a constant velocity defined using proper distances:

$$\Delta X = \int_{t_0}^{t_1} \frac{c}{a(t)} dt \tag{2.18}$$

We also have:

$$\Delta X = \int_{t_0+\delta t_0}^{t_1+\delta t_1} \frac{c}{a(t)} dt \tag{2.19}$$

Choosing δt_0 and δt_1 small compared to other time scales, we can derive from the above equations:

$$\frac{c}{a(t_0)} dt_0 = \frac{c}{a(t_0)} dt_0 \tag{2.20}$$

$$dt_1 = \frac{a(t_1)}{a(t_0)} dt_0 \tag{2.21}$$

If we think in terms of an electromagnetic wave and we accept that the number of oscillations n in the wave train does not change between emission and reception, we have to accept that the frequency has changed since they are received over a different time interval. Then we have the relation $n = dt_0 \frac{c}{\lambda_0} = dt_1 \frac{c}{\lambda_1}$, or:

$$\boxed{\boldsymbol{\lambda_1 = \frac{a(t_1)}{a(t_0)} \lambda_0}} \tag{2.22}$$

We see that wavelength transforms the same way as distance. Obviously the energy of the photon (or the associated electromagnetic field) decreases with cosmic expansion. Where does the energy go? Is energy conservation broken? Let us emphasize that this result is established in a situation where the metric remains homogeneous and isotropic and so is of limited validity. However, it is consistent with the results found when computing the covariant divergence of the stress-energy tensor of radiation in an FLRW metric, which is the form in which the energy conservation is expressed in GR.

2.2.4 Momentum decay in a flat FLRW universe

Can we characterize the motion of a single free-falling particle in a FLRW metric? From GR, we know that such a particle moves along a geodesic with equation:

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\ \nu\kappa}\frac{dx^\nu}{du}\frac{dx^\kappa}{du} = 0, \tag{2.23}$$

where u is any parameter describing the position along the geodesic (proper time for example). Let us recall that the metric connections can be derived from the metric tensor using the relation:

$$\Gamma^\mu_{\ \nu\kappa} = \frac{1}{2}g^{\mu\lambda}\left[g_{\nu\lambda,\kappa} + g_{\lambda\kappa,\nu} - g_{\nu\kappa,\lambda}\right], \tag{2.24}$$

where the comas denote a partial derivative with respect to the space-time variable corresponding the index following the coma. Since the FLRW metric is homogeneous, only time derivatives will produce non-zero results in the above formula. Moreover, since g is diagonal the two other indexes in the non-zero connection coefficients should be equal. Thus we only have to evaluate $\Gamma^0_{\ \alpha\alpha}$ and $\Gamma^\alpha_{\ 0\alpha}$. It is trivial to check that $\Gamma^0_{\ 00} = 0$. Using the usual convention that latin indices run over space coordinates only we get:

$$\Gamma^0_{\ ii} = \frac{\dot{a}a}{c}$$

$$\Gamma^i_{\ 0i} = \frac{\dot{a}}{ca}$$

Injecting these connection coefficients into the space component of the geodesic equation we get:

$$\frac{d^2x^i}{du^2} + \Gamma^i_{0i}\frac{cdt}{du}\frac{dx^i}{du} + \Gamma^i_{i0}\frac{dx^i}{du}\frac{cdt}{du} = 0 \tag{2.25}$$

We now choose u to be the proper time.

$$\frac{d^2x^i}{d\tau^2} = -2\frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^i}{d\tau} \tag{2.26}$$

This is easily integrated to show that $\frac{dx^i}{d\tau} = \frac{cst}{a^2}$. The $\frac{dx^i}{d\tau}$ are the space components of the comoving 4-velocity. Then the *space components* of the *comoving* 4-momentum are $m\frac{dx^i}{d\tau}$. It follows that the space components of the *physical momentum* evolve as:

$$p \propto a^{-1} \tag{2.27}$$

Once again, on a superficial level, conservation of momentum seems to be broken if a is not constant. However, classical mechanics is not able to describe gravitation arising from a uniform, infinite non-empty space (Poisson's equation does not apply).

2.3 The Einstein equation in the FLRW metric

To decide how the FLRW metric changes with time (through the function $a(t)$) we have first to compute the Einstein tensor associated with our metric, then model the matter content of the universe, that is specify the stress-energy tensor, and finally relate the two using Einstein's equation.

2.3.1 Computation of the Einstein tensor

The Riemann tensor is related to the metric connections through:

$$R^\alpha_{\ \beta\gamma\delta} = \Gamma^\alpha_{\ \beta\delta,\gamma} - \Gamma^\alpha_{\ \beta\gamma,\delta} + \Gamma^\mu_{\ \beta\delta}\Gamma^\alpha_{\ \mu\gamma} - \Gamma^\mu_{\ \beta\gamma}\Gamma^\alpha_{\ \mu\delta} \tag{2.28}$$

Given the small number of non-zero connection coefficients, and that only $R^{\alpha}{}_{\beta\alpha\gamma}$ are needed to compute the Ricci tensor, we can check that the only non-zero relevant Riemann tensor coefficients are:

$$R^{0}{}_{i0i} = \frac{\ddot{a}a}{c^2} \tag{2.29}$$

$$R^{i}{}_{0i0} = -\frac{\ddot{a}}{c^2 a} \tag{2.30}$$

$$R^{j}{}_{iji} = \frac{\dot{a}^2}{c^2} \qquad (i \neq j) \tag{2.31}$$

Consequently the Ricci tensor, defined by $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$, can be calculated:

$$R_{\mu\nu} = \begin{pmatrix} -3\frac{\ddot{a}}{c^2 a} & 0 & 0 & 0 \\ 0 & \frac{\ddot{a}a+2\dot{a}^2}{c^2} & 0 & 0 \\ 0 & 0 & \frac{\ddot{a}a+2\dot{a}^2}{c^2} & 0 \\ 0 & 0 & 0 & \frac{\ddot{a}a+2\dot{a}^2}{c^2} \end{pmatrix} \tag{2.32}$$

The corresponding Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu} = -\frac{6}{c^2}\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)$ and the Einstein tensor, defined by $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, is:

$$G_{\mu\nu} = \frac{1}{c^2}\begin{pmatrix} 3\frac{\dot{a}^2}{a^2} & 0 & 0 & 0 \\ 0 & -2\ddot{a}a - \dot{a}^2 & 0 & 0 \\ 0 & 0 & -2\ddot{a}a - \dot{a}^2 & 0 \\ 0 & 0 & 0 & -2\ddot{a}a - \dot{a}^2 \end{pmatrix} \tag{2.33}$$

We can check that this expression of the Einstein tensor is indeed spatially homogeneous and isotropic.

2.3.2 The stress-energy tensor in cosmology

Deciding on the content of the universe is part of the modelling. It can reflect reality more or less accurately. We will start with the most simple case beyond an empty universe, the case of *dust*.

The case of dust

In cosmology *dust* refers to a collection of particles without motions relative to each other. If we consider a local inertial frame in which dust is at rest with density ρ_0, the only non-zero component of the stress-energy tensor will be $T^{00} = \rho_0 c^2$. This level of modelling is adapted for a first approach and will already give some relevant insight on the dynamics of the universe. It will however be useful to refine it by considering a perfect fluid rather than dust: that is a case where particles do have motion relative to each other, but do no interact microscopically. For example radiation can by described as a perfect fluid but not as dust. The first step toward writing the stress-energy tensor for a perfect fluid is to write it for dust that has a velocity v (4-velocity $\mathbf{u} = \gamma(c, v_1, v_2, v_3)$) in the local inertial frame. Referring to eq. 2.4, T^{ij} is the flux of p_i through a unit surface $x_j =$ cst. The number of particles going trough the unit surface per unit time is $\frac{\rho v_j}{m_p}$ where ρ is the (relativistic) density of the moving dust and m_p is the mass of the particle. Then the flux of $p_i = \gamma m_p v_i$ is $T^{ij} = \gamma\rho v_j v_i$. Due to length contraction under Lorentz transformation, we have the relation $\rho = \gamma\rho_0$ where ρ_0 is the mass density of the dust in the local frame where it is at rest. Then $T^{ij} = \gamma^2\rho_0 v_j v_i = \rho_0 u^j u^i$. A similar approach can be followed for components involving time, thus we obtain the expression of the stress-energy tensor for dust of rest-frame density ρ_0 moving with a uniform 4-velocity $\mathbf{u}$:

$$T^{\mu\nu} = \rho_0 u^{\mu} u^{\nu} \tag{2.34}$$

The case of a perfect fluid

The particles of a perfect fluid can be considered as a collection of subsets of particles all moving with a 4-velocity within $d\mathbf{u}$ of the same 4-velocity $\mathbf{u}$ in the local rest frame of the fluid. Each of these subsets can be modelled as dust. If n is the number density of particles in the local rest frame of the fluid, and $\eta(\mathbf{p})d\mathbf{p}^3$ the number density of particles with 4-momentum whose space components are within $d\mathbf{p}^3$ of $\mathbf{p} = m_p\mathbf{u}$, the stress-energy tensor of the corresponding subset is:

$$dT^{\mu\nu} = \frac{\eta(\mathbf{p})}{\gamma}\frac{p^\mu p^\nu}{m_p}d\mathbf{p}^3 \tag{2.35}$$

Note the γ factor due to the fact that we are not using the number density in the subset's rest frame but in the fluid rest frame. The total stress-energy tensor is then:

$$T^{\mu\nu} = \int \frac{\eta(\mathbf{p})}{\gamma}\frac{p^\mu p^\nu}{m_p}d\mathbf{p}^3 \tag{2.36}$$

A first remark is that under the assumption that η only depends on the norm of $\mathbf{p}$ (true only in the local rest frame of the fluid) and is therefore an even function of the components of $\mathbf{p}$, as well as that η goes to zero fast enough at infinity, it is easy to check that T is diagonal.

Let us first focus on the space-space components of $T^{\mu\nu}$. We will show that they correspond to the relativistic pressure. In special relativity, Newton's second law states: $\mathbf{F} = \frac{d\mathbf{p}}{d\tau}$, where the vectors are 4-vectors and τ is the proper time. If the force is measured by an observer in the rest frame of the fluid, the proper time is equal to the coordinate time. Thus, the relativistic pressure can be defined by $P\vec{u} = S^{-1}\frac{d\vec{p}}{dt}$, where $\vec{p}$ is the relativistic 3-momentum of a system of surface S ($\vec{u}$ is a unit vector normal to the surface) on which the gas pressure acts. Using Newton's third law, the variation of (relativistic) momentum of the system is equal to the variation of the momentum of the gas particles hitting the surface. Without loss of generality, we can choose $\vec{u}$ along the x axis. Then, supposing the collision is elastic, the change in momentum of a particle hitting S is $-2\gamma m_p v_1 = -2p_1$. Counting the number of particles hitting the surface in dt with momentum $\mathbf{p}$, we get $Sdt\eta(\mathbf{p})\frac{p_1}{\gamma m_p}d\mathbf{p}^3$. Then, computing the momentum transfer for for all collisions within dt, the pressure is:

$$P = \int_{p_1>0} 2p_1\eta(\mathbf{p})\frac{p_1}{\gamma m_p}d\mathbf{p}^3 \tag{2.37}$$

Since the integrand is an even function of p_1 we can remove the factor 2 and integrate over the whole p-space, recovering the right hand side of eq. 2.36. Thus we showed that:

$$T^{11} = P \tag{2.38}$$

Of course, the same reasoning applies for the other space-space components: $T^{22} = T^{33} = P$. On the other hand:

$$T^{00} = \int \eta(\mathbf{p})\frac{E^2}{\gamma m_p c^2}d\mathbf{p}^3 = \int \eta(\mathbf{p})\gamma m_p c^2 d\mathbf{p}^3 = \rho c^2 \tag{2.39}$$

where ρ is the relativistic density of the gas. Finally, the stress-energy tensor of an ideal gas in the local rest frame of the gas is:

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{2.40}$$

This expression can be directly adapted to the case of a gas of photons. It can be generalized to the case where the local frame is not the rest frame of the gas:

$$T^{\mu\nu} = \left(\frac{P}{c^2} + \rho\right)u^\mu u^\nu - \eta^{\mu\nu}P \tag{2.41}$$

and then, in General Relativity to:

$$\boxed{\boldsymbol{T^{\mu\nu} = \left(\frac{P}{c^2} + \rho\right) u^\mu u^\nu - g^{\mu\nu} P}} \tag{2.42}$$

In the framework of the cosmological principle it is sufficient to consider the case where $\mathbf{u} = (c, 0, 0, 0)$. Indeed, considering a non uniform 3-velocity field of the gas would break homogeneity and isotropy and a uniform velocity field can be set to zero by choosing the correct frame. Then, the covariant form of the stress-energy tensor $T_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta}$ is written:

$$T_{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & a^2 P & 0 & 0 \\ 0 & 0 & a^2 P & 0 \\ 0 & 0 & 0 & a^2 P \end{pmatrix} \tag{2.43}$$

2.4 The Friedmann equations

Introducing in Einstein's equation the expressions of Einstein's tensor and the stress-energy tensor for an ideal gas in the framework of the cosmological principle yields two independent equations, the so-called Friedmann equations. In the case of a flat universe we can use eq. 2.33 and eq. 2.43 and we get:

$$3\frac{\dot{a}^2}{a^2} = 8\pi G \rho \tag{2.44}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -\frac{8\pi G}{c^2} P \tag{2.45}$$

Quite obviously an equation of state for the gas is needed to close this system of equations. In other words, the expansion of the universe proceeds differently depending on the nature of its content (radiations, relativistic gas, non-relativistic gas, etc.).

2.4.1 Proper (physical) and comoving distances

When discussing the distance between two points, dx is called the comoving distance between them, while $ds = a(t)dx$ is called the proper distance. Which one matches the usual notion of physical distance? The answer comes from considering that a photon travelling from one point to another and back will take the same time (as measured by an observer located at the departure point) at different epochs in the history of the universe if the physical distance between the points remains constant (as c is a constant). The time measured by the observer is $dt = \frac{a(t)dx}{c} = \frac{ds}{c}$. Then obviously, the proper distance matches the usual notion of physical distance. But it also means that the physical distance between two points with fixed (x, y, z) coordinates changes with times, thus the idea of an expanding (contracting) universe. If however we compute the space component of the relative 4-velocity between these two points, we find 0! It's very important to realize that the increased physical distance between two points with fixed spatial coordinates does not translate into an actual relative velocity. If it did, by taking two points sufficiently separated we would obtain a velocity larger than c. This pseudo-velocity is often called the Hubble flow. As it sometimes mimics the effects of the physical velocity (producing a Doppler shift for example) it is unfortunately often confused with a real velocity, even among astronomers (but usually not among cosmologists).

We will now study several simple cases for the content of the universe that are relevant for cosmology.

2.4.2 The dust universe

In section 1.3.2, we defined dust as a collection of particles without motion relative to one another. The important factor was that the relative physical velocities were zero, not that the relative physical distances remained fixed. Thus, in the framework of an expanding universe, dust will be defined as a collection of particles with fixed **comoving** relative distances (or fixed (x, y, z) coordinates). Obviously dust is characterized by $P = 0$.

Matter conservation

How does the density ρ evolves in time? Let's multiply both sides of 2.44 by a^3 and derive with respect to t:

$$\frac{d}{dt}(8\pi G\rho a^3) = \frac{d}{dt}(3\dot{a}^2 a) = 6\ddot{a}\dot{a}a + 3\dot{a}^3 = 3\dot{a}a^2\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = -3\dot{a}a^2\frac{8\pi G}{c^2}P = 0 \tag{2.46}$$

This can be rewritten in a more friendly way as:

$$\boxed{\rho(t) = \frac{\rho_0}{a^3(t)}} \tag{2.47}$$

We can understand this relation quite simply. By definition, the comoving number density (and thus density and energy density) of dust is constant. Since the physical volume is related to the comoving volume by $ds^3 = a^3(t)dx^3$, the (physical) density evolves as a^{-3}. The universe was denser when the expansion factor was smaller. This is an expression of matter conservation in a dusty universe. It can also be derived using the zero divergence of the stress-energy tensor.

2.4.3 The Einstein - de Sitter universe

The flat dusty universe, although the simplest, is of particular interest for cosmologists because i) it correctly describes the period when dark matter (behaving mostly as dust) dominated the energy content of the universe and ii) it has a simple analytic solution.

By convention, if t_0 is the present time, $a(t_0) = 1$.

Then, taking ρ_0 to be the current density of the universe and using $a = 0$ as the origin of the time axis, it is easy to solve eq. 2.44. The solution is:

$$a(t) = \sqrt{6\pi G\rho_0}\; t^{\frac{2}{3}} \tag{2.48}$$

The expanding nature of the universe and the existence of a Big Bang is already encoded in this simplest possible modelling of a non-empty universe. Something that Einstein had trouble accepting at first!

Can we compute the age of the universe if we assume the dust model was valid all along? ρ_0, the current average density of the universe, is not easy to measure. But we can make use of another observation: Hubble's law.

Computing the Hubble constant: the incorrect way

The quick and dirty way to relate the Hubble constant to the expansion factor is to treat the receding velocity of galaxies as a physical velocity: it is actually caused by the expansion of the universe. Then, considering that the fixed coordinate distance between us and a distant galaxy is Δx, the physical distance is $L = a(t)\Delta x$ and the velocity is $v = \dot{a}(t)\Delta x$. We can plug these expressions into Hubble's law:

$$v = H_0 L \tag{2.49}$$
$$\dot{a}(t_0)\Delta x = H_0 a(t)_0 \Delta x \tag{2.50}$$
$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \tag{2.51}$$

While popular, this demonstration sets the dangerous habit of considering the Hubble flows as a physical velocity and may lead to incorrect computations when large distances (with photon travel time of the order of H_0^{-1}) are involved.

Computing the Hubble constant: the correct way

Let us member that the receding velocities of galaxies are actually computed from a Doppler shift with $\frac{v}{c} = \frac{\delta\lambda}{\lambda}$. We have learned that the wavelength of a travelling photon behaves exactly as a physical distance, it increases proportionally to $a(t)$. Thus:

$$v = \frac{\delta\lambda}{\lambda}c = \frac{\delta a}{a}c = \frac{\dot{a}}{a}c\delta t = \frac{\dot{a}}{a}L \tag{2.52}$$

Let us mention that this computation does not depend on the specific time in the history of the universe when the photon is received and is valid both for the Hubble constant (referring to the present time) and for the so-called Hubble parameter (referring to any time).

$$\boxed{\boldsymbol{H(t) = \frac{\dot{a}}{a}}} \tag{2.53}$$

Computing the Hubble constant for the Einstein - de Sitter cosmology gives: $H_0 = \frac{2}{3t_0}$. Thus, the observation of the local universe allows us to estimate the age of the universe. Using the current estimation $H_0 \sim 70$ km.s^{-1}.Mpc^{-1}, yields $t_0 = 9.4$ Gyr. Einstein - de Sitter cosmology is not the model that agrees best with observations (e.g. it lacks a contribution from dark energy), but it still captures the correct orders of magnitude.

2.4.4 The case of curved space-time

We have seen that the cosmological principle does not imply that space-time is flat. A uniform positive or negative curvature is also a possibility. Deriving Friedmann's equation in those cases is somewhat more cumbersome, but still manageable. If k is the sign of the curvature (with value 1 or -1), the Friedmann's equations for a dust universe take the form:

$$3\frac{\dot{a}^2}{a^2} + 3c^2\frac{k}{a^2} = 8\pi G\rho \tag{2.54}$$
$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + c^2\frac{k}{a^2} = -\frac{8\pi G}{c^2}P \tag{2.55}$$

It is easy to check that the relation $\rho(t) = \rho_0 a^{-3}$ holds again in this case, if we still consider a dust universe ($P = 0$). It is also possible to verify that taking the time derivative of the first equation yields the second. Thus, in the case of the dust universe, the dynamical content of the second Friedmann equation disappears.

Although there exist no analytical solution to Friedmann's equations if $k \neq 0$, some properties can be derived from the form of the equation. Isolating the time derivative of the expansion factor we get:

$$\dot{a} = \sqrt{\frac{8\pi G\rho_0}{3a} - kc^2} \tag{2.56}$$

Knowing that the universe is currently expanding, we can distinguish three behaviours:

- **If k < 0 (open, saddle-like universe):** as $a(t)$ grows, $\dot{a}$ decreases asymptotically towards a positive constant. The universe will expand indefinitely.
- **If k = 0 (flat universe):** this is the Einstein - de Sitter case. The universe expands indefinitely, as $\dot{a}$ asymptotically reaches zero.
- **If k > 0 (closed, hyperspherical universe):** the universe expands, but the expansion rate reaches 0 for a finite value of $a = a_1$. Using the other possible sign in front of the square root when writing $\dot{a}$, it is possible to connect the expanding solution with a contracting solution starting at a_1. Then the universe ends in a *Big Crunch*.

2.4.5 The cosmological constant and dark energy

At the time when Einstein built GR, the prevalent view was that the universe was (on large scales) in a stationary state. With the basic version of Eintein's equation, there are no $a(t) = cst$ solution to Friedmann's equation. Consequently, Einstein introduced the simplest possible change to his equation that would yield a stationary universe: $G^{\mu\nu} - \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$ (the sign in front of Λ depends on the chosen signature), where Λ is the so-called cosmological constant. It is tempting to put the cosmological constant term on the other side of the equation and interpret it as a contribution to the stress-energy tensor. Yielding to temptation, we introduce **dark energy** as a component of the content of the universe. It acts as a uniform negative pressure. It can be shown that, in the presence of the cosmological constant, the Newtonian limit of GR yields the following modified Poisson equation: $\nabla^2\phi = 4\pi G\rho - \Lambda c^2$, where ϕ is the Newtonian potential. Though not obvious from Einstein's equation, the dark energy also acts as a negative uniform mass. Friedmann's equations can be derived as:

$$3\frac{\dot{a}^2}{a^2} + 3c^2\frac{k}{a^2} - c^2\Lambda = 8\pi G\rho \tag{2.57}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + c^2\frac{k}{a^2} - c^2\Lambda = -\frac{8\pi G}{c^2}P \tag{2.58}$$

We can check that a universe with uniform positive curvature gives $a(t) = cst$ for the correct fined-tuned value of Λ. Although the cosmological constant was dropped in the 30's when it became undeniable from the observation that the universe was expanding, it came back in the 70's, with a value not fine-tuned to yield a stationary universe, in the form of dark energy. Indeed, particle physics theory showed the particle - antiparticle pairs were constantly appearing and annihilating even in empty space, giving it an energy. This theory received observational confirmations in the 2000's from Super Novae observations that implied a $a(t)$ function only compatible with a non-zero Λ.

Once again, even in the zero-curvature case, there is not analytical solution to Friemann's equation with non-zero Λ. It is, however, easy to see that when dark energy dominates over matter, which happens at large enough a, Friedmann's equations reduces to: $\frac{\ddot{a}}{a} = \frac{c^2}{3}\Lambda$, producing an exponential, ever accelerating expansion. At small a, the cosmological constant term is negligible and the Einstein - de Sitter solution is valid.

2.4.6 The radiation universe

The universe does not contain matter only, but also radiation. Do they dominate the energy content in different situations or do we have to take both into account simultaneously (thus invalidating the previously studied solutions) ? Let's first study the case of a flat universe filled with radiation only and zero cosmological constant. If we call ϵ the radiation energy density, and P the radiation pressure, it is easy to check that, in a 1D kinetic theory toy-model, $P = \epsilon$. A full 3D theory yields $\epsilon = 3P$. We

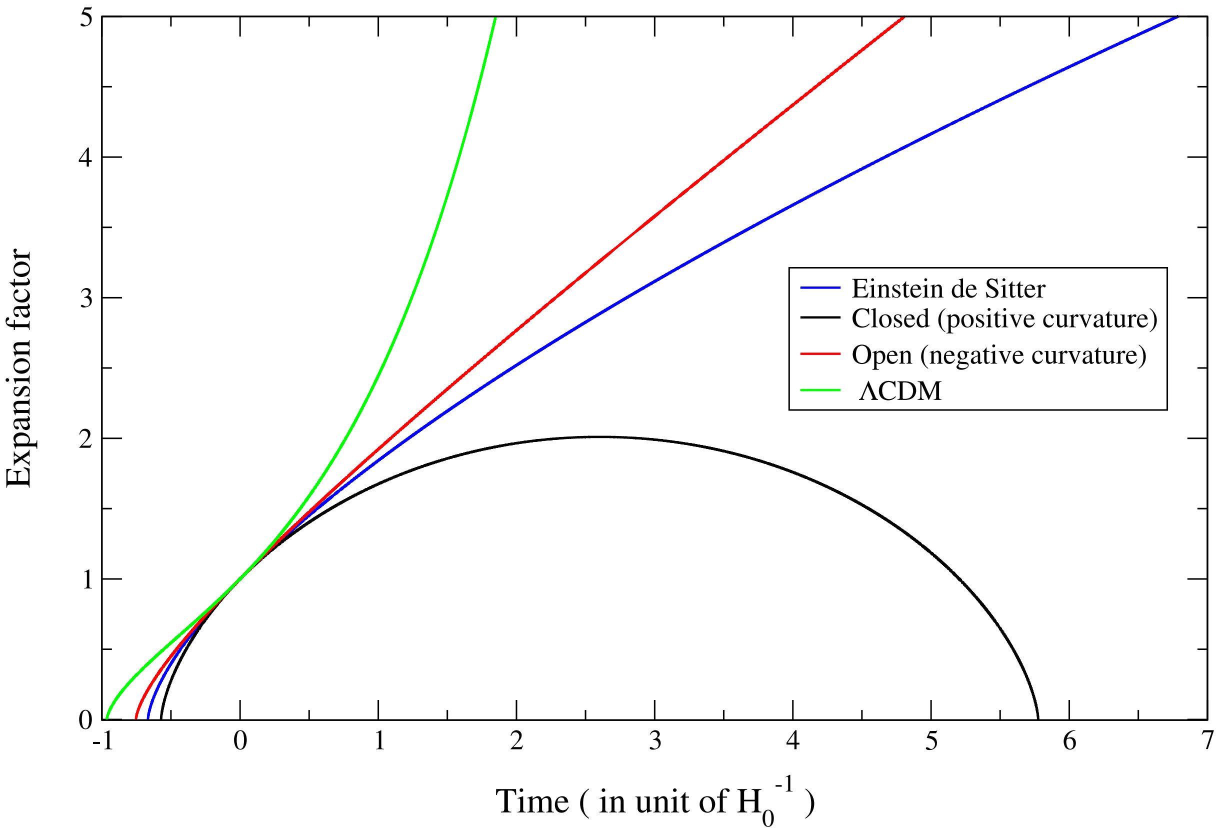

Figure 2.1: Evolution of the expansion factor as a function of time for different models of the universe. The time origin is fixed at the present time and all models satisfy the same value for the Hubble constant.

can revisit energy (instead of matter) conservation in this case. Let's multiply both sides of the first Friedmann equation by a^4 and take the time derivative (here ρ is replaced by $\frac{\epsilon}{c^2}$):

$$\begin{aligned}
\frac{d}{dt}\left(\frac{8\pi G}{c^2}\epsilon a^4\right) &= \frac{d}{dt}\left(3\dot{a}^2a^2\right) \\
&= 6\dot{a}\ddot{a}a^2 + 6\dot{a}^3 a \\
&= 3\dot{a}a^3\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) + 3\dot{a}^3 a \\
&= 3\dot{a}a^3\left(-\frac{8\pi G}{c^2}P + \frac{\dot{a}^2}{a^2}\right) \\
&= 3\dot{a}a^3\frac{8\pi G}{c^2}\left(-P + \frac{\epsilon}{3}\right) \\
&= 0
\end{aligned}$$

Thus we find that $\epsilon(t) \propto a^{-4}$. This is consistent with the view the number density of photons is proportional to a^{-3} (same as dust) while their individual energy decreases as a^{-1} (wavelength proportional to a). While rough evaluations from observations show that the energy content of the universe is currently dominated by matter over radiation (and actually dark energy over matter), at sufficiently small a, that is early enough in the history of the universe, it was dominated by radiation because of the a^{-4} behaviour of the radiation energy density. During this radiation dominated period, it is a simple matter to show from the above equations that the solution for a(t) is:

$$a(t) \propto t^{\frac{1}{2}} \tag{2.59}$$

2.5 The standard ΛCDM model of the universe

A realistic model of the universe has to include both matter and radiation as well as dark energy, and should not make any a priori assumption about the curvature. Let us first say a few words about the mixture of matter and radiation.

Mixture of dust and radiation and conservation laws

For the kinetic pressure of baryons to be non negligible compared to the rest mass energy, relativistic velocities have to be involved. On cosmological scales this happens only in the very early universe, and even then the radiation pressure was much stronger than the baryon kinetic pressure. Thus we will restrict this discussion to the case of a mixture of dust and radiation. Let us denote the matter energy density $\rho_m c^2$, the radiation energy density $\epsilon = \rho_r c^2$ and the radiation pressure $p = \frac{1}{3}\epsilon$. We have seen that when dust and radiation are considered separately, $\rho_m \propto a^{-3}$ and $\epsilon \propto a^{-4}$. We obtained this result using the two Friedmann equations and the equation of state (3 equations, 3 variables: a, ρ, P). Replacing ρc^2 by $\rho_m c^2 + \epsilon$ and P by $\frac{1}{3}\epsilon$ in Friedmann's equations, we have to solve for three variables (a, ρ_m, ϵ) with 2 equations. We need another one. This equation should describe how dust and radiation interact: for example how radiation pressure acts on dust.

While it does not provides a third independent equation, it is interesting to consider the first principle of thermodynamics. Consider a volume V. Under the assumption of the cosmological principle it will not exchange heat with the rest of the universe (heat flows would break isotropy), thus the first principle is written:

$$dU = -PdV \tag{2.60}$$

$$d(\rho_m c^2 V + \epsilon V) = -\frac{1}{3}\epsilon dV \tag{2.61}$$

If we assume that dust and radiation do not interact, this equation is replaced by $d(\rho_m V) = 0$, and $d(\epsilon V) = -\frac{1}{3}\epsilon dV$, which can be easily integrated to $\rho_m V = cst$ and $\epsilon V^{\frac{4}{3}} = cst$. Thus we generalize the $\rho_m \propto a^{-3}$ and $\epsilon \propto a^{-4}$ relations to the case of the mixture of non-interacting dust and radiation. If they do interact these relations no longer hold, and eq. 2.61 shows how matter and radiation can exchange energy density. Radiation pressure can oppose the expansion of matter. Further physical modelling is needed to quantify the efficiency of the interaction. However, all estimates show that the energy exchanged between matter and radiation is negligible compared to their respective energy, except during the very early history of the universe. **Thus we will assume from now on that dust and radiation do not interact and that the relations $\rho_m \propto a^{-3}$ and $\epsilon \propto a^{-4}$ hold.**

Parametrization of the standard cosmological model

In the general case the first Friedmann equation can be written:

$$3\frac{\dot{a}^2}{a^2} + 3c^2\frac{k}{a^2} - c^2\Lambda = 8\pi G\rho_m + 8\pi G\rho_r$$

It is relevant to introduce the quantity,

$$\boxed{\boldsymbol{\rho_c = \frac{3H_0^2}{8\pi G},}} \tag{2.62}$$

where H_0 is the Hubble constant (present value of the Hubble parameter $H(t)$). ρ_c is called the critical density of the universe (we will see why below). Then the Friedmann equation can to transformed to:

$$\frac{1}{H_0^2}\frac{\dot{a}^2}{a^2} = -\frac{c^2 k}{H_0^2 a^2} + \frac{c^2\Lambda}{3H_0^2} + \frac{\rho_m}{\rho_c} + \frac{\rho_r}{\rho_c} \tag{2.63}$$

And finally we come to the formulation:

$$\boxed{H^2 = H_0^2 \left[\Omega_\Lambda + \frac{\Omega_k}{a^2} + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} \right]} \tag{2.64}$$

with

$$\boxed{\Omega_\Lambda = \frac{c^2 \Lambda}{3H_0^2}, \qquad \Omega_k = -k\frac{c^2}{H_0^2}, \qquad \Omega_m = \frac{\rho_m(t_{\mathrm{now}})}{\rho_c}, \qquad \Omega_r = \frac{\rho_r(t_{\mathrm{now}})}{\rho_c}.} \tag{2.65}$$

Writing eq. 2.64 at $t = t_{\mathrm{now}}$ we obtain the constraint : $\Omega_\Lambda + \Omega_k + \Omega_m + \Omega_r = 1$. In a universe with negligible radiation content and without a cosmological constant, $\Omega_k = 1 - \Omega_m$. That is to say, if the matter density of the universe is currently larger than the critical density, the universe is closed (has a positive curvature and ends in a Big Crunch) and if the current matter density of the universe is less than the critical density, the universe is open (negative curvature, infinite expansion). Hence the naming of the critical density.

The five parameters $H_0, \Omega_m, \Omega_\Lambda, \Omega_k, \Omega_r$ give a first level modelling of the universe. For a more precise description, parameters describing how homogeneity is broken on small scales (e. g. amplitude of density fluctuations) are necessary. When trying to constrain these parameters from observations (of the CMB for example), various minimal sets of parameters are used depending on the observation. Combining several types of observations (CMB, weak lensing, Baryon acoustic oscillations, etc.), the current favoured model is:

Parameter	Value
H_0	67.74 ± 0.46 km.s^{-1}.Mpc^{-1}
Ω_m	0.3089 ± 0.0062
Ω_Λ	0.6911 ± 0.0062
Ω_k	0.0005 ± 0.0006
Ω_r	$9.10^{-5} \pm 3.610^{-6}$

A first conclusion is that a flat universe is currently favoured. Even if it is not actually flat, curvature was never a driving factor for the expansion. The second conclusion is that radiation is currently negligible.

3 The cosmic microwave background (CMB)

As we mentioned in the introduction the CMB is, from an observational point of view, an isotropic emission that very accurately fits the spectrum of a black-body at 2.7 K. The bulk of the universe is currently transparent to the typical wavelength of this black-body. However we learned that wavelength scales with the expansion factor. This raises two questions: what was the shape of the spectrum at an earlier time, and what happened when it interacted with matter?

3.1 Black-body radiation in a expanding universe

A black-body spectrum at temperature T is characterized by a photon number density per frequency interval $d\nu$:

$$n(\nu, T)d\nu = \frac{8\pi\nu^2}{c^3}\frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}d\nu \tag{3.1}$$

At a different epoch, the frequency of a photon is changed to $\nu_1 = \frac{a}{a_1}\nu$ and the photon number density at that frequency is $n_1(\nu_1)d\nu_1 = \frac{a^3}{a_1^3}n(\nu)d\nu$ (the comoving number density of photons within $d\nu$ is unchanged but is now covering a $d\nu_1$ frequency interval). Thus,

$$\begin{aligned} n_1(\nu_1)d\nu_1 &= \frac{a^3}{a_1^3}\frac{8\pi\nu^2}{c^3}\frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}d\nu \\ &= \frac{8\pi\nu_1^2}{c^3}\frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}d\nu_1 \\ &= \frac{8\pi\nu_1^2}{c^3}\frac{1}{\exp\left(\frac{h\nu_1}{k_B T\frac{a}{a_1}}\right) - 1}d\nu_1 \end{aligned}$$

This is just a black-body distribution with temperature $T_1 = T\frac{a}{a_1}$. As long as we ignore the interaction between matter and radiation, the CMB evolves as a black-body with a temperature:

$$\boxed{\boldsymbol{T \propto a^{-1}}} \tag{3.2}$$

Current radiation density and the redshift of equivalence

From the spectral number density of photons for a black-body, it is a simple matter of integrating over the frequency to derive the radiation energy density. This gives the well known formula:

$$\epsilon = \frac{4\sigma T^4}{c} \qquad \text{with} \qquad \sigma = \frac{2\pi^5 k_B^4}{15h^3c^2} \quad \text{the Stefan} - \text{Boltzmann constant} \tag{3.3}$$

We can now compute the current contribution of the CMB photons to the critical density of the universe using the black-body temperature $T_{\text{cmb}} = 2.7$ K :

$$\Omega_r^{\text{cmb}} = \frac{\rho_r}{\rho_c} = \frac{\epsilon}{c^2\rho_c} = \frac{4\sigma T_{\text{cmb}}^4}{c^3}\frac{8\pi G}{3H_0^2} \sim 5.\,10^{-5} \tag{3.4}$$

As we can see, the CMB is contributing about half of the best-estimate value of $\Omega_R \sim 9.\,10^{-5}$. Other ultra relativistic particles such as neutrinos contribute to the remaining part of Ω_R.

As we already mentioned in the section on the ΛCDM model, the current radiation energy density is negligible compared to both the matter energy density and the dark energy density. The relative contributions were different in the past, however. Eq. 2.64 shows the radiation contribution scales at a^{-4}, the matter contribution as a^{-3} and the dark energy contribution is constant (to the best of our knowledge). Thus there was a time when the contribution of radiation and energy were equal. The corresponding redshift is called the *redshift of equivalence* and is denoted z_{eq}. It can be simply computed as:

$$z_{eq} = \frac{\Omega_m}{\Omega_r} - 1 \sim 3400 \tag{3.5}$$

Then, the universe was $\sim 10^{10}$ times denser than now! Before that time, we can assume that the universe was radiation dominated ($a \propto t^{\frac{1}{2}}$), then matter dominated ($a \propto t^{\frac{2}{3}}$) and finally dark energy dominated ($a \propto \exp(c\sqrt{\frac{\Lambda}{3}}t)$). Assuming no coupling between the CMB and matter (an assumption that needs to be verified at high redshift) we can also estimate simply the temperature of the CMB at equivalence:

$$T_{\text{cmb}}(z_{eq}) = T_{\text{cmb}}(z=0)(1+z_{eq}) \sim 9000\text{K} \tag{3.6}$$

Recombination

We have already mentioned that there should be a period in the history of the universe when the gas, cooling with expansion, should recombine, going from a plasma to a collection of neutral atoms. Later on, the primordial photons (not produced in stars and such) are effectively decoupled from matter. When does this happen? Before or after the redshift of equivalence? We will now estimate the temperature when recombination occurred.

The process of recombination relies on the following reversible reaction:

$$\mathbf{p}^+ \quad + \quad \mathbf{e}^- \quad \rightleftharpoons \quad \mathbf{H} \quad + \quad \gamma \tag{3.7}$$

We will consider that the time scale associated with this photo-ionization/recombination process is much shorter than the Hubble time H^{-1} (this is no longer true when the fraction of free electrons drops to $\sim 10^{-4}$) and that the different species are in thermo-dynamical equilibrium. We will consider a set of N electron and N protons (to ensure a global nul-charge), and we will call n the number of free electrons and protons. Then the number of hydrogen atoms is $N - n$. This set of particles (and each species subset) easily exchanges energy with a heat bath through the ionizing/recombination photons. The CMB itself is the heat bath (the energy density of the CMB is much greater than the kinetic and binding energy of the particles). And thus we assume that our set of particles is in thermal equilibrium with the heat bath at temperature T. Then, a single, non-interacting particle of type i (i denoting either electron, proton or atom) has the following partition function:

$$z_i = \int g_i \exp\left(-\frac{m_i c^2 + \frac{p^2}{2m_i}}{k_B T}\right) \frac{d^3x}{h^3} d^3p, \tag{3.8}$$

where g_i is the multiplicity of each microscopic state, m_i the mass of the particle and p its momentum. Notice that we include the rest mass energy. Considering that the particles occupy a volume V, we can transform this integral to:

$$z_i = \frac{4\pi g_i V}{h^3} (k_B T 2 m_i)^{\frac{3}{2}} \exp\left(-\frac{m_i c^2}{k_B T}\right) \int_O^\infty x^2 e^{-x^2} dx$$

Using the result $\int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$ we get:

$$z_i = \frac{g_i V}{h^3} (k_B T 2\pi m_i)^{\frac{3}{2}} \exp\left(-\frac{m_i c^2}{k_B T}\right) \tag{3.9}$$

If we consider the particles to be non-interacting and indistinguishable, the partition function for electrons is $\frac{z_e^n}{n!}$, and for the full set of particles:

$$Z = \frac{z_e^n}{n!} \frac{z_p^n}{n!} \frac{z_H^{N-n}}{(N-n)!} \tag{3.10}$$

The quantity n is an internal variable for this system. Then if $F = -k_B T \ln(Z)$ is the free energy of the system, the equilibrium value of n is such that $\frac{\partial F}{\partial n} = 0$. This yields the relation:

$$\frac{\partial \ln(Z)}{\partial n} = \frac{\partial}{\partial n} \left[n \ln(z_e) + n \ln(z_p) + (N-n)\ln(z_H) - 2\ln(n!) - \ln(N-n)\right] = 0$$

Injecting the Stirling formula $\ln(n!) \sim n \ln n - n$ we get:

$$\begin{aligned} \ln\left(\frac{z_e z_p}{z_H}\right) &= \frac{\partial}{\partial n}\left[2n\ln(n) - 2n + (N-n)\ln(N-n) - N - n\right] \\ &= 2\ln(n) - \ln(N-n) \\ \frac{z_e z_p}{z_H} &= \frac{n^2}{N-n} \end{aligned}$$

If we denote the ionization fraction of the gas$x = \frac{n}{N}$, and we plug in the expression of the partition functions:

$$\frac{x^2}{1-x} N = \frac{g_e g_p}{g_H} \frac{V (k_B T 2\pi)^{\frac{3}{2}}}{h^3} \left(\frac{m_e m_p}{m_H}\right)^{\frac{3}{2}} \exp\left(-\frac{m_e c^2 + m_p c^2 - m_H c^2}{K_B T}\right) \tag{3.11}$$

Now, electrons and protons have spin 1/2 so $g_e = g_p = 2$ and the ground state of hydrogen has two hyperfine levels with spin 0 and 1 with respective multiplicity 1 and 3, so $g_H = 4$. We make the approximation $m_p = m_H$, denote the ionization energy of Hydrogen $E_0 = m_e c^2 + m_p c^2 - m_H c^2$, and the number density of all hydrogen species (neutral and ionized) $n_B = N/V$, the above equation simplifies to:

$$\frac{x^2}{1-x} = \frac{(2\pi m_e k_B T)^{\frac{3}{2}}}{n h^3} e^{-\frac{E_0}{k_B T}} \tag{3.12}$$

This relation is known as the Saha equation. Obviously if T is large enough x goes to 1 (fully ionized plasma) and if T goes to zero x goes to zero (fully neutral gas). If the factor in front of the exponential was of the order one, recombination would occur around a temperature that assures $k_B T \sim E_0$, that

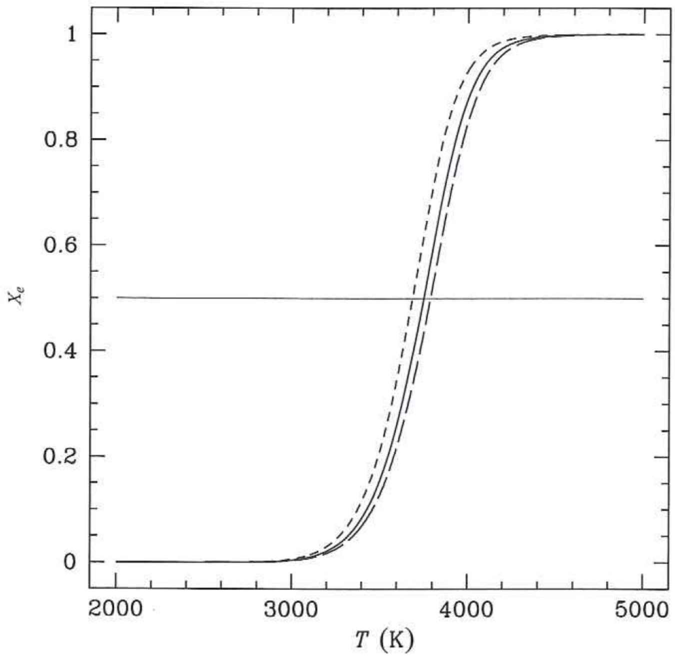

Figure 3.1: Behaviour of the ionization fraction of hydrogen as a function of temperature for different value of the baryon number density corresponding, from left to right to $\Omega_b = 0.02$, 0.04 and 0.06. Figure taken from "The Cosmic Microwave background", 2008, R. Durrer, Cambridge Univesity Press.

is $T \sim 100\,000$K. However, in actuality the factor is large and the transition occurs at a significantly lower temperature. Moreover, we see that a larger number density of baryons induces recombination at a larger temperature (and thus an earlier epoch). The function $x(T)$ is shown in fig. 5.2. We can see that the universe recombined within a rather narrow range of temperatures between 4000 K and 3000 K (redshift $z \sim 1300$, about 300000 years after the Big Bang). By that time, matter was already driving the expansion of the universe.

4

Inflation

4.1 Evolution of the Universe curvature (the flatness coincidence)

Does the curvature of the Universe change with time? Let's go back to the first Friemann equation:

$$3\frac{\dot{a}^2}{a^2} + 3c^2\frac{k}{a^2} - c^2\Lambda = 8\pi G(\rho_m + \rho_r) \tag{4.1}$$

Let us define the critical density at *any time* t as $\rho_c(t) = \frac{3H^2(t)}{8\pi G}$. Then we can define the cosmological parameters at *any time* t:

$$\Omega_k(t) = -\frac{kc^2}{H^2(t)a^2(t)}, \qquad \Omega_\Lambda(t) = \frac{c^2\Lambda}{3H^2(t)} \qquad \Omega_m(t) = \frac{\rho_m(t)}{\rho_c(t)}, \qquad \Omega_r(t) = \frac{\rho_r(t)}{\rho_c(t)} \tag{4.2}$$

In terms of these quantities, the Friedmann equation can be rewritten:

$$\Omega_k(t) = 1 - \Omega_\Lambda(t) - \Omega_m(t) - \Omega_r(t) \tag{4.3}$$

This relation tells us how the curvature of the Universe evolves in time. We can see that a flat universe ($k = 0$) remains flat since the curvature term drops out of the Friedmann equation. Let us now consider the case where $|\Omega_k(t_{\text{now}})| \sim 1$. We will ignore the contribution from Ω_Λ (but including it does not present any difficulty nor change the results much). Let us fist evaluate Ω_k at the time of equivalence t_{eq}. After t_{eq} we have:

$$\Omega_k(t) = -\frac{kc^2}{H^2(t)a^2(t)} = -\frac{kc^2}{\dot{a}^2(t)} \propto t^{\frac{2}{3}} \propto a(t) \tag{4.4}$$

Since $a(t_{eq}) \sim 3.10^{-4}$, we must have $\Omega_k(t_{eq}) \sim 3.10^{-4}$. Before t_{eq}, during the radiation dominated era we have:

$$\Omega_k(t) = -\frac{kc^2}{\dot{a}^2(t)} \propto a^2(t) \tag{4.5}$$

Let us remember that by radiation, we mean ultra-relativistic particles of any nature. In the standard model of particle physics, particles can exist in the form of quarks up to temperatures of 10^{12} K (~ 100 MeV). Thus, moving toward the Big Bang, a radiation-dominated universe description is reasonable at least up to this point (beyond that, unification of the fundamental forces begins). Since in a radiation dominated universe $a \sim T^{-1}$, we can establish that:

$$\frac{a_{\text{rad}}}{a_{\text{eq}}} = \frac{T_{\text{eq}}}{T_{\text{rad}}} \tag{4.6}$$

where the $_{\rm rad}$ subscript designates quantities evaluated at the beginning of the radiation-dominated era. Thus:

$$\Omega_k(t_{\rm rad}) = \Omega_k(t_{\rm eq}) \left(\frac{a_{\rm rad}}{a_{\rm eq}}\right)^2 = \Omega_k(t_{\rm eq}) \left(\frac{T_{\rm rad}}{T_{\rm eq}}\right)^{-2} \sim 10^{-20} \tag{4.7}$$

Let us emphasize that this already small value is more or less a maximum because we chose a conservative value for $T_{\rm rad}$. Another estimate pushes the validity of the radiation dominated era up to $T_{\rm rad} \sim 10^{28} K$, the onset of the Grand Unification Theory. In any case we find that, in a universe dominated by matter or radiation, the curvature increases with time. Therefore, in order to arrive at a Ω_k not much larger than 1, we must start with a vanishingly small value. This is called the **asymptotic flatness problem**. While not an inconsistency of the theory in and of itself, this requires a fine tuning of the parameters (here initial curvature) that physicists do not like much. Unless the curvature was driven to such a low value at the begin of the radiation-dominated era by a previous regime during which the curvature was decreasing.... A decreasing Ω_k implies an increasing $\dot{a}$, or, in other words, a period of **accelerating expansion**. Such a behaviour is provided, for example, by the cosmological constant which creates an asymptotically exponential expansion. Some similar component must have dominated the very early universe.

4.2 The CMB isotropy paradox

The CMB is observed to be isotropic (once various contaminations have been removed) to one part in 10^4. This means that the slice of universe revealed by the CMB was homogeneous at this level in both temperature and density. This is believable only if it was able to reach and maintain homogeneity trough some process (pressure forces for example). Then, it is necessary that a signal travelling at the speed of light (at best) had time to travel across this slice of the universe in the time interval between the Big Bang and when the CMB was emitted.

Particle horizon at recombination

The comoving distance covered between time t_0 and t by a particle travelling at the speed of light is:

$$\Delta x = \int_{t_0}^{t} \frac{c}{a(t)} dt \tag{4.8}$$

If we choose $t_0 = 0$, we define the particle horizon h_p as the maximum comoving distance between two particles so that, at time t, they have been able to interact (exchange photons, neutrinos, feel the other's gravitational pull, etc) since the Big Bang. Changing the integration variable, we can write:

$$\begin{aligned} h_p &= \int_0^a \frac{c}{\dot{a}(t)a} da \\ &= \int_0^a \frac{c}{H(a)} \frac{da}{a^2} \\ &= \int_z^\infty \frac{c}{H(z)} dz \end{aligned}$$

In a flat ΛCDM model we get:

$$h_p = \frac{c}{H_0} \int_z^\infty \frac{dz}{\sqrt{\Omega_\Lambda + \Omega_m(1+z)^3 + \Omega_r(1+z)^4}} \tag{4.9}$$

Assuming a totally radiation dominated universe until z_{eq} and a totally matter dominated universe between z_{eq} and z_{rec} allows us to estimate the integral analytically with 20% accuracy. A numerical integration using the ΛCDM numerical value for the parameters yields $h_p = 0.055\frac{c}{H_0}$. We want to compare this with the comoving radius of the spherical slice of the universe where we observe the CMB. This radius is determined by the distance travelled by a photon between z_{rec} and now:

$$h_{cmb} = \frac{c}{H_0} \int_0^{z_{rec}} \frac{dz}{\sqrt{\Omega_\Lambda + \Omega_m(1+z)^3 + \Omega_r(1+z)^4}} \tag{4.10}$$

Once again, we could get an order-of-magnitude estimation (10% accuracy) by considering a matter dominated universe. The correct numerical computation gives $h_{cmb} = 3.145\frac{c}{H_0}$. As we can see, the comoving particle horizon at z_{rec} is small compared to the comoving size of the spherical slice that we observe. We can even estimate that only regions of angular size less than $\frac{h_p}{h_{cmb}} \sim 1\,\text{deg}$ were causally connected. It appears then, that a simple ΛCDM model cannot explain the homogeneity of the CMB.

Requirement of the expansion law for solving the paradox

If we want to solve the horizon problem, we have to modify our model in such a way that $h_p > h_{cmb}$. The easiest way to do this is to assume that the energy content of the every early universe obeys a different equation of state compared to that of radiation.

It is useful (and usual) to described a generic, one-component universe with the parameterized equation of state $P = w\rho c^2$. Then $w = 0$ corresponds to the dust universe, $w = 1/3$ describes the radiation universe and $w = -1$ a universe where dark energy dominates. The Friedmann equations can be solved easily in this generic case. The solution is $a(t) = \left(\frac{t}{t_{\text{now}}}\right)^{\frac{2}{3(1+w)}}$ if $w \neq -1$, and $a(t) = e^{Ht}$ (where H does not depend on time) if $w = -1$.

Now, what value of w yields a solution to the horizon problem? Let's compute h_p for a generic expansion law $a(t) \propto t^\alpha$ where $\alpha = \frac{2}{3(1+w)}$. Since $H = \alpha a^{-1/\alpha}$ in this case, we get:

$$h_p = \frac{c}{\alpha} \int_{z_{req}}^{\infty} (1+z)^{-\frac{1}{\alpha}} dz \tag{4.11}$$

It is interesting to see that if $\alpha > 1$ this integral diverges. This case corresponds to an accelerating expansion ($\ddot{a} \propto \alpha(\alpha-1)t^{\alpha-2} > 0$). So the same condition that solved the asymptotic flatness problem seems to help with the isotropy paradox. $\alpha > 1$ also means $w < -\frac{1}{3}$, a condition that is not compatible with ordinary matter and radiation but is satisfied by dark energy for example. If we assume that for a given period in the history of the very early universe this condition was satisfied, by pushing the redshift of the beginning of this period to a high enough value, we can make h_p larger than h_{cmb} and solve the horizon problem.

This period of accelerated expansion is called inflation. Let us now look into what kind of energy content may source the inflation.

4.3 A scalar field as the source of inflation

As inflation occurred at energies where not even quarks exist, beyond the domain of validity of the standard model of particle physics, there are few constraints on the nature of the field that creates it. Consequently, the minimal model is to consider a self-interacting scalar field, called the inflaton. In GR, its lagrangian density can be written:

$$\mathcal{L}_\phi = \frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu \phi - V(\phi) \tag{4.12}$$

4.3.1 Stress-energy tensor of a scalar field

In the general case, there are two methods of deriving the form the the stress-energy tensor from the expression of the Lagrangian (density). The first is to consider it as the non-gravitational part of the least-action principle applied to the action $S = \int \left[\frac{c^4}{16\pi G} R + \mathcal{L}_\phi\right] \sqrt{-\det(g)} dx^4$. This yields the relation

$T^{\mu\nu} = -2\frac{\delta \mathcal{L}_\phi}{\delta g_{\mu\nu}} + g_{\mu\nu}\mathcal{L}$ (where δ is used to designate a functional derivative). The second approach is to consider the stress-energy tensor as the Noether current associated with the conservation of energy and momentum, that is associated with the invariance of the action under space-time translations. In the case of a scalar field, the stress-energy tensor takes the form:

$$T_{\mu\nu} = \partial_\mu\phi\;\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - V(\phi)\right) \tag{4.13}$$

We now want to simplify this expression in the case of a FLRW metric. To be consistent with the cosmological principle, the scalar field can ony depend on time, not on the spatial coordinates. The first consequence is that all off-diagonal terms in $T_{\mu\nu}$ are zero. Then we can easily check that:

$$\begin{aligned} T_{00} &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ T_{jj} &= \frac{1}{2}a^2\dot{\phi}^2 - a^2V(\phi) \end{aligned}$$

By analogy with the case of a perfect fluid we can define the energy density and pressure of the scalar field:

$$\rho_\phi c^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{4.14}$$

$$P_\phi = = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{4.15}$$

4.3.2 Evolution of the scalar field: the Klein-Gordon equation

In the case of a perfect fluid, the equation governing the evolution of $\rho(t)$ can be established either directly from the Friedmann equations or more meaningfully by writting the nul-divergence condition for the energy momentum tensor. It takes the form (established in the exercise session): $\dot{\rho}c^2 = -3H(t)(\rho c^2 + P)$. The same derivation holds in the case of a scalar field. Injecting the expressions for ρ_ϕ and P_ϕ, we get the equation driving the evolution of ϕ, the Klein-Gordon equation:

$$\boxed{\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0} \tag{4.16}$$

where V' is the derivative of $V(\phi)$ with respect to ϕ. If we want to formulate a point-particle dynamics analogy, the field is subject to a force derived from the potential V and a drag force from expansion.

4.3.3 Conditions for inflation to occur

We established that $w < -\frac{1}{3}$ is a necessary condition to solve the CMB isotropy paradox, and that Ω_k should decrease by a factor of at least 10^{20} and possibly much larger. Choosing $w = -1$ produces an exponential evolution of the expansion factor that easily produces large modifications of the curvature. However, injecting P_ϕ and ρ_ϕ in the equation of state shows that only a frozen scalar field is compatible ($\dot{\phi} = 0$), and thus inflation will never end. So let us instead consider a near-exponential inflation created by a scalar field obeying $w = -1 + \alpha$ with $\alpha \ll 1$. In this case we have

$$\begin{aligned} P_\phi &= (-1+\alpha)\rho_\phi c^2 \\ (1-\alpha/2)\dot{\phi}^2 &= \alpha V(\phi) \\ \dot{\phi}^2 &\ll V(\phi) \end{aligned}$$

The kinetic energy of the inflaton must be much smaller than its potential energy. This is called **slow roll inflation**. Under this condition, the first Friedmann equations simplifies to:

$$H^2 = \frac{8\pi G}{3}V(\phi) \tag{4.17}$$

Now, we also want this condition to hold for a large number of Hubble times, to give inflation enough time to impact the curvature and the particle horizon. A way to express this is to say that the time scale on which the ratio $\frac{\dot{\phi}^2}{V(\phi)}$ changes is much longer than H^{-1}. That is:

$$\left|\frac{V(\phi)}{\dot{\phi}^2}\frac{d}{dt}\left(\frac{\dot{\phi}^2}{V(\phi)}\right)\right| \ll H$$

$$\left|\frac{2\ddot{\phi}}{\dot{\phi}} - \dot{\phi}\frac{V'}{V}\right| \ll H$$

Injecting the Klein-Gordon equation to remove V':

$$\left|\frac{2\ddot{\phi}}{\dot{\phi}} + \frac{\dot{\phi}\ddot{\phi}}{V} + 3H\frac{\dot{\phi}^2}{V}\right| \ll H$$

Using the slow roll condition, this reduces to the condition:

$$\left|\frac{\ddot{\phi}}{\dot{\phi}}\right| \ll H \tag{4.18}$$

Under this condition the Klein-Gordon equation simplifies to $3H\dot{\phi} + V'(\phi) = 0$. Using the simplified Klein-Gordon and Friedmann equations, it is simple to rewrite the slow roll and extended expansion conditions in the form

$$\boxed{\left|\frac{V'}{V}\right| \ll \sqrt{24\pi G} \qquad \text{and} \qquad \left|\frac{V''}{V}\right| \ll 8\pi G} \tag{4.19}$$

To obtain these relations, it is useful to establish first, from the simplified Friedmann and Klein-Gordon equations that $\dot{H} = -4\pi G\dot{\phi}^2$. To summarize, these relations guaranty that, for a given value of the inflaton (i.e. at a given time), the expansion is nearly exponential and will remain so for a large number of Hubble times.

4.3.4 A simple case of self-interacting scalar field

A familiar case of self-interacting field is the case of a scalar field driven by a mass term: $V(\phi) = \frac{1}{2}m^2\phi^2$. Then the two conditions for inflation reduce to a similar condition:

$$\phi \gg \frac{1}{\sqrt{4\pi G}} \tag{4.20}$$

Inflation persists as long as the value of the inflaton is (much) larger than the Planck scale. From the simplified Klein-Gordon equation we know that during near-exponential inflation, the inflaton value decreases near-exponentially. Thus the above condition also sets the time scale for the end of inflation.

5

Using supernovae as cosmic candles

5.1 Standard candles

Dying stars

During their life, stars are supported against their own gravity by thermal pressure. Since their thermal energy is radiated away as light, they need an energy source to maintain their pressure: the thermonuclear fusion of hydrogen into Helium in the core of the star (where the temperature is sufficiently high). Toward the end of their life, hydrogen is completely consumed, pressure drops and the core contracts gravitationally. It becomes denser and hotter until the thermonuclear fusion of heavier elements (helium, carbon, neon, oxygen, silicon...) becomes possible, restoring thermal pressure. If the mass of the star is less than 8 $M_\odot$ the core will not reach sufficiently high temperature and density to fuse carbon because the collapse will be stopped before that point by the Fermi pressure of electrons. These stars shed their outer layers and a core remains with mass $< 1.4\ M_\odot$: a white dwarf. More massive stars cannot be supported by the Fermi pressure of electrons. They will contract further and convert the matter in their core into Fe. Fe is the end product of thermonuclear fusion, the most stable element: energy cannot be gained by turning it into something else. Then fusion will switch off in the core, the pressure will drop, and the star will collapse on itself triggering a so-called type II core collapse supernovae. The exploding star (outer layers rebound on the core) reach a luminosity comparable to that of a full galaxy during a few weeks, making it visible at cosmological distances. In the case of type II supernovae however, the peak luminosity varies with the mass of the star.

What would be useful for cosmology is a standard candle: an object of known luminosity. Measuring the received flux would then gives us its distance. If receding velocity is also measured by Doppler shifts we have two quantities that are related through the expansion law and thus we can test our cosmological model against observations. Galaxies are not standard candles either: their luminosity dependsnor just on their mass which is already very difficult to determine independently, but also on their stellar population, morphological type, etc... Type Ia supernovae, on the other hand are reasonably close to being standard candles.

Type Ia supernovae

The progenitor of a Type Ia supernovae is a binary system where a red giant and a white dwarf orbit each other. As the envelope of the red giant expands the material reaches of Roche limit and is sucked down by the white dwarf, whose mass gradually increases. When it reaches 1.4 $M_\odot$, the Fermi pressure of electrons cannot support the star any longer and it undergoes gravitational collapse followed by a thermonuclear explosion, forming a type Ia supernovae. All these, originating in stars with 1.4 $M_\odot$ mass, have similar luminosity. Some variations occur depending on the metal content of the progenitor star, but those variations can be estimated using information from the spectrum of the supernovae. They are therefore good standard candles to use for cosmology.

5.2 The brightness-distance relation

Assuming that absorption is negligible along the line of sight, the flux received from a supernovae can be computed as a function of its redshift if we assume that the supernovae follows the Hubble flow, that is has no peculiar velocity. We start with the relation between the comoving distance to the supernovae r_c and its redshift:

$$r_c = \int_0^z \frac{c}{H(z)} dz \tag{5.1}$$

Then the received flux is:

$$E(z) = \frac{L_{SN}}{4\pi r_c^2}(1+z)^{-2} \tag{5.2}$$

Two effects contribute to the $(1+z)^{-2}$ factor. First, the photons emitted in a time interval dt are received in $(1+z)dt$ (see section 2.2.3). Then their energy has decreased by a factor $(1+z)^{-1}$. A usual quantity to measure the received flux is the apparent magnitude $m(z) = -2.5 \log\left(\frac{E(z)}{E_0}\right)$, where E_0 is a reference flux (that of Vega). But in the case of supernovae, we can even get rid of the reference flux and supernovae luminosity by computing the ratio of received flux, or difference of magnitudes. We will compare the supernovae to a reference supernovae located 10 pc away from us, because this matches the definition of the absolute magnitude M:

$$r(z) = m(z) - m(z_{10\text{pc}}) = m(z) - M = -2.5 \log\left(\frac{E(z)}{E(z_{10\text{pc}})}\right) \tag{5.3}$$

For example, in an Einstein - de Sitter universe ($\Omega_m = 1$), we have $r_c(z) = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)$, and evaluating $z_{10\text{pc}} = \frac{H_0 \times 10^{-5}}{c} \sim 2.25\,10^{-9}$ using the Hubble law, we find the relation.

$$r(z) = 44.75 + 5 \log\left[\left(1 - \frac{1}{\sqrt{1+z}}\right)(1+z)\right] \tag{5.4}$$

While this formula is insensitive to the value of Ω_m, it is valid only for $\Omega_m = 1$. In the general case, with $\Omega_\Lambda \neq 0$, $r(z)$ is sensitive to both the values of Ω_m and Ω_Λ. The $r(z)$ can be computed for any cosmological model, and can be plotted from observation using spectroscopic measurement for z and bolometric ones for $r(z)$. Thus, waiting long enough to observe many supernovae, as wall as observing faint ones to probe large z values, we can test cosmological models.

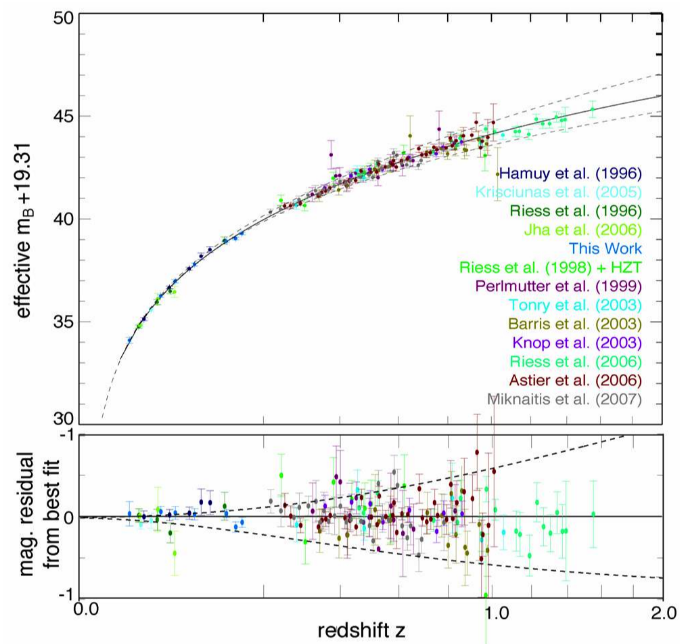

Figure 5.1: Comparison of SN data to different cosmological models (adapted from Kowalski, 2008, ApJ, 686,749). The solid line is the best fit ΛCDM model. The bottom dashed line is for the Einstein - de Sitter model.

6 Writing classical physics equations in a flat, homogeneous, expanding metric

Cosmological models can only be tested against observations. Observable quantities, usually objects emitting light, are caused by a number of physical processes: hydrodynamics, radiative transfer, magnetic fields, etc... In RG, the equation governing these processes are derived, in the general case, by building a relevant stress energy tensor and then writing $T^{\alpha}_{\beta;\alpha} = 0$. In some cases it is possible to use a linear perturbation theory around an isotropic/homogeneous metric (for describing the CMB anisotropies for example). In many other cases it is even simpler. The metric is not even perturbed: only perturbations for the variables for hydrodynamics, radiative transfer and other equation describing physical processes are considered. This regime is usually called Newtonian perturbation theory.

6.1 Regime of validity

The rigorous method for determining when the Newtonian perturbation theory is valid is to write the covariant version of the equation describing the physical processes ($T^{\alpha}_{\beta;\alpha} = 0$), build a perturbation theory around an homogeneous cosmological metric including perturbation of the metric, apply it to $T^{\alpha}_{\beta;\alpha} = 0$ and Einstein's equation, and finally search for the perturbation parameters (several "small" parameters may be needed: small velocities, weak gravitational fields, etc...) that allow us to recover minimally modified versions of the classical equations. This is beyond the scope of this document, but can be found in many cosmological textbooks for the case of Euler's equation.

Let's us, however, state what the necessary conditions are under which the Newtonian perturbation theory is valid. A first obvious condition is that the perturbation to the metric and stress-energy tensor should be small. In the case of dust, this means that the density perturbations should remain small. But to be able to revert to Newtonian gravity in an expanding homogeneous space-time and study only the density perturbation, there is another condition. The size of the regions showing a departure from the average properties of the universe (or the wavelength of the modes considered) should be much smaller than the Hubble Radius $R_H = \frac{c}{H}$. Indeed, if this is not the case, by the time the information of a growing density patch has travelled to the other side of the region, the expansion factor has changed significantly: a situation that can obviously not be handled using Newtonian gravity (no retarded potentials) in a homogeneously expanding space-time.

6.2 Rewriting physical quantities and the differential operators in comoving space

Classical physics equations are valid using physical (not comoving) quantities: proper distance, proper time, etc... In a flat space-time, choosing a frame where matter is at rest on average, we can replace the proper time with the coordinate time. Using physical distances as a system of coordinates is, however, inconvenient. If we do so, we already need a non-zero velocity field to describe the unperturbed expansion (the Hubble Flow). A more practical way is to write the equation in comoving space. We will now outline a procedure to do so. Let's denote $\mathbf{r}$ as the proper (physical) distance, $\mathbf{x}$ as the comoving distance, $\mathbf{v_r}$ as the physical velocity and $\mathbf{v_x}$ as the comoving velocity (some textbooks use the peculiar velocity av_x instead as a variable). We have the relations:

$$\mathbf{r} = a(t)\mathbf{x} \tag{6.1}$$

$$\mathbf{v_r} = \dot{a}\mathbf{x} + a\mathbf{v_x} \tag{6.2}$$

Although the Hubble flow (first term of the r.h.s of eq. 6.2) should not be considered an actual velocity, it does appear in the relation between $\mathbf{v_r}$ and $\mathbf{v_x}$. This would be a problem if it could take values of the order of, or larger than, c. But the condition that $\mathbf{x} \ll R_H$ ensures than $\dot{a}\mathbf{x} \ll c$. Converting spatial partial derivatives is simple enough:

$$\frac{\partial}{\partial r_i} = \frac{1}{a}\frac{\partial}{\partial x_i} \tag{6.3}$$

Although proper and coordinates time are identical in our case, the time partial derivative requires a more careful examination. Indeed, we want to express the relation between $\frac{\partial}{\partial t}\big|_{\mathbf{r}}$ and $\frac{\partial}{\partial t}\big|_{\mathbf{x}}$. Let's write the differential of $f(t, \mathbf{r})$:

$$\begin{aligned}
df &= \left.\frac{\partial f}{\partial t}\right|_{\mathbf{r}} dt + \frac{\partial f}{\partial r_1}dr_1 + \frac{\partial f}{\partial r_2}dr_2 + \frac{\partial f}{\partial r_3}dr_3 && (6.4)\\
&= \left.\frac{\partial f}{\partial t}\right|_{\mathbf{r}} dt + \frac{\partial f}{\partial r_1}d(ax_1) + \frac{\partial f}{\partial r_2}d(ax_2) + \frac{\partial f}{\partial r_3}d(ax_3) && (6.5)\\
&= \left.\frac{\partial f}{\partial t}\right|_{\mathbf{r}} dt + \frac{1}{a}\frac{\partial f}{\partial x_1}(x_1 da + a dx_1) + \frac{1}{a}\frac{\partial f}{\partial x_2}(x_2 da + a dx_2) + \frac{1}{a}\frac{\partial f}{\partial x_3}(x_3 da + a dx_3) && (6.6)\\
&= \left(\left.\frac{\partial f}{\partial t}\right|_{\mathbf{r}} + \frac{\dot{a}}{a}(\mathbf{x}.\nabla_{\mathbf{x}})f\right) dt + \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 && (6.7)
\end{aligned}$$

Thus by identification:

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{r}} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} - H(\mathbf{x}.\nabla_{\mathbf{x}}) \tag{6.8}$$

6.3 Continuity equation

In physical space the continuity equation is written:

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{r}} + \nabla_{\mathbf{r}}.(\rho\mathbf{v_r}) = 0 \tag{6.9}$$

Using the comoving density $\rho_{\mathbf{x}} = \rho a^3$ we can cast this equation in comoving space:

$$\left.\frac{\partial \rho_{\mathbf{x}}\, a^{-3}}{\partial t}\right|_{\mathbf{x}} - H(\mathbf{x}.\nabla_{\mathbf{x}})(\rho_{\mathbf{x}}\, a^{-3}) + \frac{1}{a}\nabla_{\mathbf{x}}\left[\rho_{\mathbf{x}}\, a^{-3}(\dot{a}\mathbf{x} + a\mathbf{v_x})\right] \quad = \quad 0 \tag{6.10}$$

Plugging in the classical vector calculus formula $\nabla.(\phi\mathbf{A}) = \mathbf{A}.\nabla\phi + \phi\nabla.\mathbf{A}$, where ϕ is a scalar and $\mathbf{A}$ a vector, the above equation reduces to:

$$\left.\frac{\partial \rho_{\mathbf{x}}}{\partial t}\right|_{\mathbf{x}} + \nabla_{\mathbf{x}}.(\rho_{\mathbf{x}}\mathbf{v_x}) = 0, \tag{6.11}$$

and thus it takes the exact same form in comoving space, using comoving variables, as in physical space. This is not the case if the peculiar velocity $a\mathbf{v_x}$ is used, nor for the Euler equation, regardless of which velocity is used.

6.4 Poisson's equation

In a universe without a cosmological constant, it can be shown that RG yields the usual Poisson's equation in the weak field limit:

$$\nabla_{\mathbf{r}}^2 \phi = 4\pi G\rho \tag{6.12}$$

where ϕ is the gravitational potential. Considering the relation between the potential and the Newtonian force, the comoving potential is naturally defined as $\phi_{\mathbf{x}} = \phi a$, and the Poisson's equation is also unchanged in comoving space:

$$\nabla_{\mathbf{x}}^2 \phi_{\mathbf{x}} = 4\pi G\rho_{\mathbf{x}} \tag{6.13}$$

If a cosmological constant is considered, in the weak field limit gravitation is governed by the modified Poisson's equation $\nabla_{\mathbf{r}}^2 \phi = 4\pi G\rho - c^2\Lambda$ that transforms into $\nabla_{\mathbf{x}}^2 \phi_{\mathbf{x}} = 4\pi G\rho_{\mathbf{x}} - c^2\Lambda a^3$ in comoving space.

6.5 Euler's Equation

Before we write Euler's equation, we should ask ourselves two questions. Is dust an adequate description of the content of the perturbed universe, and is Euler's equation adequate to describe dust? Since we are considering perturbations around an homogeneous universe, we cannot assume that the (comoving) relative distances between particles will remain fixed. However, the important property of dust is that we can use $P = 0$. In practice, as long as $P \ll \rho c^2$ (that is, as long as the velocities are non relativistic and radiation pressure is negligible), we can describe the content of the universe as dust. Whether dust ca be described as a fluid is a different issue. The main component of matter is dark matter, composed of particles that interact with each other and with baryons only through gravitation. The short range interactions that are necessary to enable a fluid description (avoiding multiple streams) are absent. A collisionless Boltzmann equation would be a proper description for such a system. However, in the regime of small perturbations arising from homogeneous cosmological initial conditions, multistream features will not have had time to develop yet and a fluid description is acceptable. In physical space, Euler's equation for a self-gravitating pressureless fluid takes the usual from:

$$\left.\frac{\partial \mathbf{v_r}}{\partial t}\right|_{\mathbf{r}} + \mathbf{v_r}.\nabla_{\mathbf{r}}\mathbf{v_r} = -\nabla_{\mathbf{r}}\phi \tag{6.14}$$

In comoving space, this transforms to

$$\left.\frac{\partial(\dot{a}\mathbf{x} + a\mathbf{v_x})}{\partial t}\right|_{\mathbf{x}} - H\mathbf{x}.\nabla_{\mathbf{x}}(\dot{a}\mathbf{x} + a\mathbf{v_x}) + (\dot{a}\mathbf{x} + a\mathbf{v_x}).\frac{1}{a}\nabla_{\mathbf{x}}(\dot{a}\mathbf{x} + a\mathbf{v_x}) = -\frac{1}{a^2}\nabla_{\mathbf{x}}\phi_{\mathbf{x}}. \tag{6.15}$$

Taking care that $\left.\frac{\partial(\dot{a}\mathbf{x}+a\mathbf{v_x})}{\partial t}\right|_{\mathbf{x}} = \ddot{a}\mathbf{x} + \dot{a}\mathbf{v_x} + a\left.\frac{\partial \mathbf{v_x}}{\partial t}\right|_{\mathbf{x}}$, Euler's equation in comoving coordinates reduces to:

$$\left.\frac{\partial \mathbf{v_x}}{\partial t}\right|_{\mathbf{x}} + 2H\mathbf{v_x} + \mathbf{v_x}.\nabla_{\mathbf{x}}\mathbf{v_x} = -\frac{1}{a^3}\nabla_{\mathbf{x}}\phi_{\mathbf{x}} - \frac{\ddot{a}}{a}\mathbf{x} \tag{6.16}$$

We can see two additional terms compared to the physical space version. $2H\mathbf{v_x}$ is a drag term created by expansion on comoving velocities. This drag term also exists, with a different numerical coefficient, if we use the peculiar velocity as a variable. We can verify that in the absence of gravity, in the linear regime, $\mathbf{v_x} \propto a^{-2}$. The other new term, on the r.h.s, is spurious in a way. Indeed, it appears to cancel the Newtonian force created at $\mathbf{x}$ by a non-zero homogeneous density background. One could argue that this force should be zero, for symmetry reasons. However, in applying Gauss theorem one can find a non-zero force directed at the origin of coordinates (wherever it is chosen!).

The point is that Poisson's equation is ill-behaved (and should not be used) for a source field whose L^2-norm is not finite. But of course this involves scales beyond R_H where we do not expect our theory to hold. Looking at density perturbations, we will subtract the average non-zero value and thus get rid of this large scale inconsistency.

7 The growth of density perturbations

7.1 Newtonian perturbation theory

Zero-order solution

The continuity equation in comoving space (eq 6.11) admits the following homogeneous solution (we drop the $_{\mathbf{x}}$ subscript from now on to denote comoving quantities):

$$\rho_0 = cst \qquad \mathbf{v}_0 = 0 \tag{7.1}$$

Integrating Poisson's equation in spherical coordinates yields $\nabla\phi_0 = \frac{4\pi G\rho_0}{3}\mathbf{x} + \frac{A}{x^3}\mathbf{x}$, where A is an integration constant. Injecting Friedman's equation to get rid of ρ_0 we get:

$$\nabla\phi_0 = -a^3\frac{\ddot{a}}{a}\mathbf{x} + \frac{A}{x^3}\mathbf{x} \tag{7.2}$$

Then, we can check that $(\mathbf{v}_0, \phi_0)$ is a solution of Euler's equation for $A = 0$.

In comoving space a homogenous solution to the continuity, Poisson and Euler's equations (eq. 6.11, 6.13 and 6.16) is:

$$\rho_0 = cst, \qquad \mathbf{v}_0 = 0, \qquad \nabla\phi_0 = -a^3\frac{\ddot{a}}{a}\mathbf{x} \tag{7.3}$$

Perturbations

Let us consider small perturbations around the zero-order solution:

$$\rho = \rho_0(1+\delta) \qquad \mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 \qquad \phi = \phi_0 + \phi_1 \tag{7.4}$$

The quantity δ is called the overdensity and is much used in structure formation theory. Note that it has the same value in physical and comoving space. Injecting these expression in the continuity, Poisson and Euler equations, the zero-order solution cancels out (as expected) and dropping 2nd order terms we get the linearised set of equations:

$$\nabla^2\phi_1 = 4\pi G\rho_0\delta \tag{7.5}$$

$$\frac{\partial\delta}{\partial t} + \nabla.\mathbf{v_1} = 0 \tag{7.6}$$

$$\frac{\partial\mathbf{v_1}}{\partial t} + 2H\mathbf{v_1} = -\frac{1}{a^3}\nabla\phi_1 \tag{7.7}$$

Taking the divergence of the linearised Euler's equation and reinjecting Poisson's and the continuity equations we finaly get:

$$\boxed{\ddot{\delta} + 2H\dot{\delta} - 4\pi G\frac{\rho_0}{a^3}\delta = 0} \tag{7.8}$$

7.2 Solutions

During the matter dominated era

During the matter dominated era in a ΛCDM universe, the expansion faction behaves as $a(t) = At^{\frac{2}{3}}$, with a constant, different from the Einstein de Sitter model. Then $H(t) = \frac{2}{3t}$, and injecting the simplified Friedmann first equation in eq. 7.8, we get:

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0 \tag{7.9}$$

The general solution to this equation is:

$$\delta(t) = At^{-1} + Bt^{\frac{2}{3}} \tag{7.10}$$

Here we see there is one decaying mode and one growing mode. Neglecting the decaying mode we get that in fact $\delta(t) \propto a(t)$.

All perturbation in the matter density field, as long as their typical scale is smaller than the Hubble length, grow linearly with the expansion factor during the matter dominated era.

During the radiation dominated era

It is not straightforward to apply the Newtonian perturbation theory during the radiation dominated era, and a fully consistent approach requires a much more complex GR perturbation theory. Let's keep going with a Newtonian perturbation theory, injecting some knowledge learned from the full GR theory when needed.

Let us first consider the case of a universe filled with matter obeying the equation of state $P = c_s^2\rho$ (typically not dust but an ideal gas). It is then easy to check that eq. 7.8 is modified in the following way:

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2}\nabla^2\delta - 4\pi G\frac{\rho_0}{a^3}\delta = 0 \tag{7.11}$$

In the case of a static universe $(H = 0, a = 1)$ we recover the equation describing the Jeans instability in a self-gravitating fluid with dispersion equation $\omega^2 = c_s^2k^2 - 4\pi G\rho_0$. In this case, the amplitude of small scale modes with $k > \frac{\sqrt{4\pi G\rho_0}}{c_s}$ will oscillate, while the amplitude of larger scale modes will grow exponentially. Pressure support sets a minimal scale (the Jeans length) for the density perturbation to be able to grow. In the case of an expanding universe, this minimal scale will change with time, and the drag imposed by the $2H\dot{\delta}$ will slow the growth to a power law rate and damp the oscillations. Although photons do not actually behave as a self-gravitating fluid, they are, in GR, subject to gravity through their energy density and supported by their pressure. Estimating the Jeans length for photons $(c_s \sim c)$ during the radiation dominated era, one finds a value much larger than the horizon. Thus we infer that the amplitude of perturbations in the photon energy density evolves with (slowly) damped oscillations. This can be shown rigorously with a full GR treatment.

From GR we know that during the radiation dominated period, radiation is the dominant component driving the expansion. This implies that in the weak field limit, Poisson's equation should include their contribution. Defining the total density $\rho = \rho_m(1+\delta_m) + \rho_r(1+\delta_r)$, where $\delta_r = \frac{\epsilon}{c^2}$ is the equivalent radiation density, the linearized Poisson equation is $\nabla^2\phi_1 = 4\pi G(\rho_m\delta_m + \rho_r\delta_r)$. As we assume that matter and radiation are not interacting except through gravitation, they obey their own separate conservation and dynamical equations. Thus the continuity and Euler equations describing matter are unchanged. Then the linearized density perturbation equation is simply modified to be:

$$\ddot{\delta}_m + 2H\dot{\delta}_m - \frac{4\pi G}{a^3}(\rho_m\delta_m + \rho_r\delta_r) = 0 \tag{7.12}$$

Since δ_r locally oscillates on a short time scale, it averages out to zero. The secular evolution of δ_m on the other hand, is driven by changes in $a(t)$ as dictated by the above equation. Then we can expect that, in the secular evolution, $\ddot{\delta}_m \sim H^2\delta_m = \frac{8\pi G}{3a^3}\rho\delta_m$. Since in the matter dominated era $\rho = \rho_r + \rho_m \gg \rho_m$, we get $\ddot{\delta}_m \gg \frac{4\pi G}{a^3}\rho_m\delta_m$. Thus, the evolution of matter perturbation on long time scales (much longer than the oscillation periods for the photon density fluctuations) during the radiation dominated era can be described with the approximate equation:

$$\ddot{\delta}_m + t^{-1}\dot{\delta}_m = 0, \tag{7.13}$$

where we have injected $a(t) \propto t^{\frac{1}{2}}$. The general solution to this equation is:

$$\delta(t) = A + B\ln(t) \tag{7.14}$$

During the radiation dominated era, matter density fluctuations with typical sizes smaller than the Hubble length grow as $\ln(a)$. Compared to a power law growth, they are nearly frozen.

7.3 The initial density perturbation field

7.3.1 Gaussian random fields

Ramdom fields

A 3D random field is a set of random variables $Y(\mathbf{x})$, one for each location (or infinitesimal cube of volume dx^3) in 3D space. Such a field is characterized by a collection of joint probability distribution functions:

$$P_n(Y(\mathbf{x_1}) = y_1, \ldots, Y(\mathbf{x_n}) = y_n) \tag{7.15}$$

The initial density fluctuation can be described as a particular type of random field.

The cosmological principle and random fields

In a very general way, we stated the cosmological principle as *the universe being homogeneous and isotropic on large enough scales.* Considering now that the initial density fluctuations in our universe, described by the overdensity δ, are simply a realisation of a random field characterized by the joint probability distribution functions P_n, we can apply the homogeneity and isotropy requirements to the distribution functions themselves. Thus we require them to be invariant under translation and rotation. For example, the one-point distribution function of the overdensity should not depend on the location.

Gaussian random fields

The simplest flavour of inflation predicts that primordial density fluctuations are actually an homogeneous Gaussian random field. In that case, the P_1 function is simply a Gaussian probability function with a variance independent of location. The values of δ at two different locations are not uncorrelated, however, and thus is it not so easy to build a realisation of initial density fluctuations in real space. It is much easier to do in Fourier space because of the following two properties:

- The Fourier transform of the Gaussian random field is a Gaussian random field. Thus the real and imaginary parts of $\hat{\delta}(\mathbf{k})$ (the Fourier transform of $\delta(\mathbf{x})$) are random variables with Gaussian probability distribution functions.
- The homogeneity requirement translates in Fourier space into the Fourier coefficients being independent realisations of the (k-dependent) probability function.

Thus a Gaussian random field is more easily described and generated in Fourier space. The only thing that we need to characterize the field is a prescription on how the variance of the Gaussian probability function depends on $\mathbf{k}$. This information is provided by the so-called power spectrum.

7.3.2 2-point correlation function and power spectrum

We mentioned that in real space, the values of a homogeneous Gaussian random field are drawn from a single probability function, but are not independent (non-zero correlation function). Alternatively in Fourier space they are independent but drawn from a Gaussian function whose variance depends on $\mathbf{k}$. Actually, the real space correlations are determined by the k-dependence of the Fourier space variance. This is quantified by the relation between the 2-point correlation function and the power spectrum.

Let us use the following convention for the direct and inverse Fourier transforms:

$$\hat{f}(\mathbf{k}) = \int f(\mathbf{x}) e^{-i\mathbf{k}.\mathbf{x}} d^3x \tag{7.16}$$

$$f(\mathbf{x}) = \frac{1}{(2\pi)^3} \int \hat{f}(\mathbf{k}) e^{i\mathbf{k}.\mathbf{x}} d^3k \tag{7.17}$$

The 2-point correlation function of the overdensity (or any quantity) is defined as:

$$\xi(\mathbf{x_1}, \mathbf{x_2}) = \langle \delta(\mathbf{x_1}) \delta(\mathbf{x_2}) \rangle \tag{7.18}$$

where $\langle \dots \rangle$ denotes the average over many realisations of the random field, also called the "ensemble average". Since ξ is completely determined by the underlying P_1 and P_2, which are invariant under translation and rotation due to the the cosmological principle, ξ is a function of $r = |\mathbf{x_2} - \mathbf{x_1}|$ only. To show the relation between ξ and the power spectrum let us insert the Fourier transform of δ into the simplified expression $\xi(r)$.

$$\begin{aligned}
\xi(r) &= \langle \delta(\mathbf{x}) \delta(\mathbf{x} + \mathbf{r}) \rangle && (7.19)\\
&= \left\langle \frac{1}{(2\pi)^6} \int \int \hat{\delta}(\mathbf{k}) e^{i\mathbf{k}.\mathbf{x}} \hat{\delta}(\mathbf{k}') e^{i\mathbf{k}'.(\mathbf{x}+\mathbf{r})} d^3k \; d^3k' \right\rangle && (7.20)\\
&= \frac{1}{(2\pi)^6} \int \int \langle \hat{\delta}(\mathbf{k}) \hat{\delta}(\mathbf{k}') \rangle \; e^{i(\mathbf{k}+\mathbf{k}').\mathbf{x}} \, e^{i\mathbf{k}'.\mathbf{r}} d^3k \; d^3k' && (7.21)
\end{aligned}$$

We define the power spectrum $P(k)$ with the relation:

$$\langle \hat{\delta}(\mathbf{k}) \hat{\delta}(\mathbf{k}') \rangle = P(k) (2\pi)^3 \delta_D(\mathbf{k} + \mathbf{k}') \tag{7.22}$$

where δ_D designates the Dirac delta function. Remembering that the real and imaginary parts of the Fourier coefficients of δ are independent realisations of a Gaussian probability function, and that since

δ is real, $\hat{\delta}(\mathbf{k})$ and $\hat{\delta}(-\mathbf{k})$ are complex conjugates, it is quite easy to show that $P(k)$ is proportional to the variance of the Gaussian probability distribution of the Fourier coefficients. Injecting this relation into 7.21 and integrating over k we get:

$$\xi(r) = \frac{1}{(2\pi)^3} \int P(k') e^{i\mathbf{k}'.\mathbf{r}} d^3k' \tag{7.23}$$

This shows that the 2-point correlation function is simply the Fourier transform of the power spectrum. Both encode the same information about the Gaussian random field.

Note that even when applied to a dimensionless quantity, $P(k)$ has a dimension of k^{-3}. Thus it is quite usual to define the "dimensionless" power spectrum $\Delta^2(k) = \frac{k^3}{2\pi^2} P(k)$. This quantity is meaningful when considering the variance of the real space signal, that can be computed as:

$$\begin{aligned}
\langle \delta(x)^2 \rangle &= \xi(0) && (7.24)\\
&= \frac{1}{(2\pi)^3} \int P(k) d^3k && (7.25)\\
&= \int \Delta^2(k)\ d\ \ln(k) && (7.26)
\end{aligned}$$

We see that Δ^2 quantifies the contribution to the variance (in some sense the energy density) of the real space signal per logarithmic bin.

7.3.3 Linear evolution of the initial density fluctuations power spectrum

Initial power spectrum

It is often stated in textbooks on cosmology and structure formation that inflation predicts a "near scale-invariant power spectrum for the initial fluctuations". This statement, although true, needs to be contextualized! It applies to the fluctuations of the gravitational potential when considering the dimensionless power spectrum Δ^2:

$$\Delta^2_\phi(k) = cst \tag{7.27}$$

Each logarithmic bin contributes equally to the variance of the gravitational potential. From this, knowing that $\hat{\delta}(k) \propto k^2 \hat{\phi}(k)$ (from Poisson's equation), we get:

$$P_\delta(k) \propto k \tag{7.28}$$

So the dimensional power spectrum of the density fluctuation is not scale free, which is sometimes a source of confusion. We have shown that the density fluctuations grow linearly with the scale factor on sub-horizon scales in the matter dominated era. That is, all corresponding Fourier modes (above a certain k) grow at the same rate. To be able to determine the evolution of the power spectrum we need more information. First, the horizon is growing, and some modes that were initially larger than the horizon will enter the horizon at some point. So we need information on the growth of super-horizon fluctuations. Then we need to know what happened during the radiation dominated era.

The growth of fluctuations on super-horizon scale

A rigorous treatment of the linear growth of fluctuations requires a full GR treatment, especially on super-horizon scales where no other theory is valid. Since such a treatment is beyond the scope of these lecture notes let us simply state the result:

- **On super-horizon scales, during the radiation dominated era, comoving density fluctuations grow as a^2.**
- **On super-horizon scales, during the matter dominated era, comoving density fluctuations grow as a.**

Let us emphasize that there is a lot of work and subtleties involved in deriving these results (starting with the fact that "comoving" in the above statement refers to the comoving gauge choice).

Evolution of the power spectrum during the radiation dominated era

Fluctuations grow as $\ln(a)$ on sub-horizon scales, much more slowly than on super-horizon scales where they grow as a^2. We will make the approximation that they are simply frozen on sub-horizon scales until the redshift of equivalence. Let us consider 2 modes, k_1 and k_2, entering the horizon at expansion factors a_1 and a_2 respectively. Until a_1 they have grown at the same rate, so the ratio of their amplitude has not changed from the initial value:

$$\frac{|\hat{\delta}(k_1, a_1)|}{|\hat{\delta}(k_2, a_1)|} = \left(\frac{k_1}{k_2}\right)^{\frac{1}{2}} \tag{7.29}$$

Between a_1 and a_2, $\hat{\delta}(k_1)$ is frozen and remains almost unchanged. On the other hand, $\hat{\delta}(k_2)$ grows as a^2. Thus:

$$\frac{|\hat{\delta}(k_2, a_2)|}{|\hat{\delta}(k_2, a_1)|} = \frac{a_2^2}{a_1^2} \tag{7.30}$$

Consequently,

$$\frac{|\hat{\delta}(k_1, a_2)|}{|\hat{\delta}(k_2, a_2)|} = \frac{a_1^2}{a_2^2}\left(\frac{k_1}{k_2}\right)^{\frac{1}{2}} \tag{7.31}$$

In the radiation dominated era, the horizon is:

$$h = \int_z^{\infty} \frac{c}{H(z)} dz \simeq \frac{c}{H_0} \int_z^{\infty} \frac{dz}{\sqrt{\Omega_r (1+z)^4}} = \frac{c}{H_0\sqrt{\Omega_r}}\ a \tag{7.32}$$

So we have the simple relation $k \propto a^{-1}$ between the comoving wave number and the expansion factor when it enters the horizon. Thus we have $\frac{a_1^2}{a_2^2} = \frac{k_2^2}{k_1^2}$. So

$$\frac{|\hat{\delta}(k_1, a_2)|}{|\hat{\delta}(k_2, a_2)|} = \left(\frac{k_1}{k_2}\right)^{-\frac{3}{2}} \tag{7.33}$$

After a_2 the two modes follow the same evolution (frozen then growing as a) and this ratio does not change. Modes that enter the horizon after radiation-matter equivalence always follows the same growth, and the ratio of their amplitudes is unchanged from its initial value.

If we define k_{eq} as the wave number of the mode that enters the horizon at the redshift of equivalence z_{eq}, we can deduce that anytime in the matter dominated era (when the linear theory remains valid), the power spectrum of matter fluctuations follows this behaviour:

Evolved linear powerspectrum:

- **If** $k \ll k_{eq}$**:** $P(k) \propto k$
- **If** $k \gg k_{eq}$**:** $P(k) \propto k^{-3}$

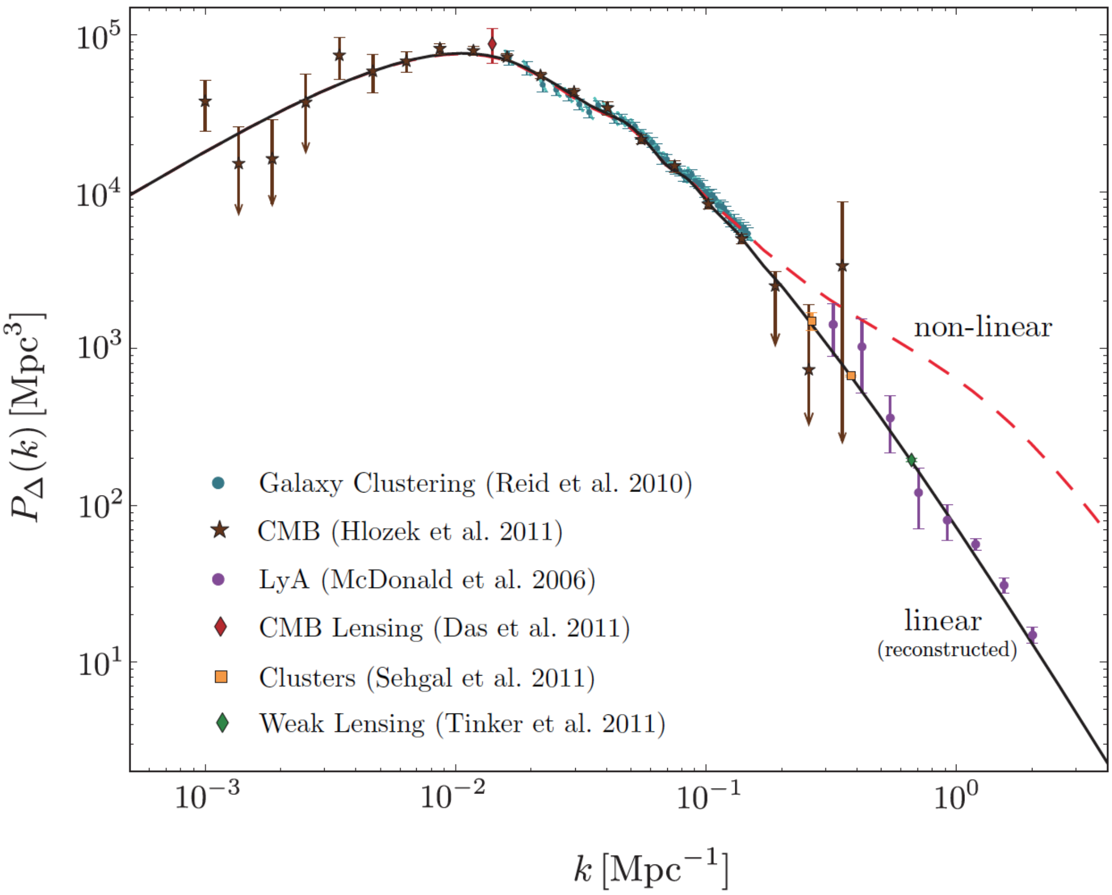

Figure 7.1: Compilation of recent observations of the matter power spectrum compared with theoretical prediction. Credit: D. Baumann, "Cosmology", Lecture notes. Reproduced with informal authorisation.