Statistical mechanics of the shallow-water system with a prior potential vorticity distribution

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Abstract

We adapt the statistical mechanics of the shallow-water equations to the case where the flow is forced at small scales. We assume that the statistics of forcing is encoded in a prior potential vorticity distribution which replaces the specification of the Casimirs constraints in the case of freely evolving flows. This determines a generalized entropy functional which is maximized by the coarse-grained PV field at statistical equilibrium. Relaxation equations towards equilibrium are derived which conserve the robust constraints (energy, mass and circulation) and increase the generalized entropy.

Résumé

Mécanique statistique d'un système à eau peu profonde avec une distribution de vorticité potentielle fixée à priori Nous adaptons la mécanique statistique d'un système à eau peu profonde au cas où le flot est forcé à petite échelle. Nous supposons que la statistique du forcing est représentée par une distribution de vorticité potentielle, qui remplace la spécification des contraintes de Casimir dans le cas des flots libres. Ceci détermine une fonctionnelle d'entropie généralisée qui est maximisée par le champ PV (vorticité potentielle) à gros grain à l'équilibre statistique. Les équations de relaxation vers l'équilibre sont obtenues, conservant les constraintes robustes (énergie, masse et circulation) et accroissant l'entropie généralisée.

Key words: 2D turbulence, geophysical flows, statistical mechanics

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1 Introduction

Two-dimensional turbulent flows with high Reynolds numbers have the striking property of organizing spontaneously into large-scale coherent structures such as jets and vortices. Jovian atmosphere shows a wide diversity of structures: Jupiter's great red spot, white ovals, brown barges,... A good hydrodynamical model to describe jovian atmosphere is provided by the Shallow-Water (SW) equations. A statistical theory of the SW system has been developed recently by Chavanis & Sommeria (2002). This extends the statistical mechanics of the incompressible 2D Euler equation proposed by Miller (1990) and Robert & Sommeria (1991) to the case of compressible flows. It is however assumed that the flow is weakly compressible (small Mach number) so that the effect of waves is not dominant. Therefore, large-scale coherent vortices can form and persist for a long time. After a phase of chaotic mixing (violent relaxation), the system is expected to reach an equilibrium state which corresponds to the most mixed state consistent with the constraints imposed by the dynamics. Mathematically, it is obtained by maximizing a mixing entropy while conserving energy, mass, circulation and all the higher moments of the PV vorticity (Casimirs). One difficulty with this approach is that the moments of PV depend on the resolution at which the PV field is considered so that the prediction can change accordingly. Furthermore, geophysical flows are usually forced and dissipated at small-scales so that the conservation of all the Casimirs is abusive. In an attempt to solve these problems, Ellis et al (2002) have proposed to fix a prior vorticity distribution instead of the Casimirs. It is assumed that this global distribution of vorticity is generated by the small-scale forcing so it must be given as an *input* in the statistical theory. In this approach, only the conservation of the robust constraints (E, Γ) is taken into account and the effect of the small-scale forcing is encapsulated in a prior vorticity distribution or in a generalized entropy. This approach has been further developed in Chavanis (2005a) and Chavanis (2005b) in the context of the quasi-geostrophic (QG) equations. In this paper, we show how it can be extended to the case of the Shallow-Water (SW) equations. Since this is only a slight, but interesting, variant with respect to the un-forced case, we shall mostly refer to the study of Chavanis & Sommeria (2002) for technical details, without repeating all the steps of the derivation.

2 The shallow-water equations

The dynamical evolution of a thin fluid layer with thickness $h(x, y, t)$ and velocity field $\mathbf{u} = (u, v)(x, y, t)$ submitted to a gravity acceleration g on a rotating planet is governed by the shallow water equations

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -g\nabla h - 2\mathbf{\Omega} \times \mathbf{u}. \tag{2}$$

The first equation can be viewed as an equation of continuity and the second equation as the Euler equation (in a rotating frame $\mathbf{\Omega}$) for a barotropic fluid with pressure $p = \frac{1}{2}gh^2$. The Euler equation can be rewritten

$$\frac{\partial \mathbf{u}}{\partial t} + (\boldsymbol{\omega} + 2\mathbf{\Omega}) \times \mathbf{u} = -\nabla B, \tag{3}$$

where $\boldsymbol{\omega} = \omega \mathbf{e}_z = \nabla \times \mathbf{u}$ is the vorticity and where we have introduced the Bernouilli function

$$B = \frac{\mathbf{u}^2}{2} + gh. \tag{4}$$

The potential vorticity (PV)

$$q = \frac{\omega + 2\Omega}{h}, \tag{5}$$

is conserved for each fluid parcel, i.e.

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0. \tag{6}$$

Each mass element $hd\mathbf{r}$ is also conserved in the course of the evolution. This implies that the shallow-water equations conserve the PV moments

$$\Gamma_n = \int q^n h d\mathbf{r}. \tag{7}$$

The moments $n = 0, 1, 2$ are, respectively, the total mass M, the circulation Γ and the PV enstrophy Γ_2. The energy

$$E = \int h \frac{\mathbf{u}^2}{2} d\mathbf{r} + \frac{1}{2}\int gh^2 d\mathbf{r}, \tag{8}$$

involving a kinetic and a potential part is also conserved. It is convenient to use a Helmholtz decomposition of the momentum $h\mathbf{u}$ into a purely rotational and purely divergent part

$$h\mathbf{u} = -\mathbf{e}_z \times \nabla\psi + \nabla\phi. \tag{9}$$

For any stationary solution, the mass conservation (1) reduces to $\nabla \cdot (h\mathbf{u}) = 0$ so that

$$h\mathbf{u} = -\mathbf{e}_z \times \nabla\psi, \tag{10}$$

where ψ is the stream-function. Then Eq. (6) reduces to $\mathbf{u} \cdot \nabla q = 0$ which implies that $q = f(\psi)$. Finally, Eq. (3) reduces to

$$(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \times \mathbf{u} = -\nabla B. \tag{11}$$

Taking the scalar product with $\mathbf{u}$, we obtain $\mathbf{u} \cdot \nabla B = 0$ so that $B = B(\psi)$. Then, substituting Eq. (10) in Eq. (11) we obtain

$$q = -\frac{dB}{d\psi}. \tag{12}$$

The SW equations admit an infinite class of stationary solutions, specified by the relations $B = B(\psi)$ and $q = f(\psi) = -B'(\psi)$. They are determined by the two coupled partial differential differential equations for ψ and h

$$-\frac{\Delta\psi}{h^2} + \frac{2\Omega}{h} + \frac{1}{h^3}\nabla\psi \cdot \nabla h = -\frac{dB}{d\psi}, \tag{13}$$

$$\frac{(\nabla\psi)^2}{2h^2} + gh = B(\psi). \tag{14}$$

This formulation of the SW equations in terms of (h, ψ) variables has been introduced in Chavanis & Sommeria (2002).

3 The equilibrium statistical mechanics

3.1 Freely evolving flows

The SW equations are known to develop a complicated mixing process which ultimately leads to the emergence of a large-scale coherent structure, typically a jet or a vortex. One question of fundamental interest is to understand and predict the structure and the stability of these equilibrium states. This can be achieved by using statistical mechanics arguments. The idea is to replace the deterministic description of the flow $q(\mathbf{r},t)$ by a probabilistic description where $\rho(\mathbf{r},\sigma,t)$ gives the density probability of finding the PV level $q=\sigma$ in $\mathbf{r}$ at time t (it satisfies the local normalization condition $\int \rho(\mathbf{r},\sigma)d\sigma = 1$). The observed (coarse-grained) PV field is then expressed as $\overline{q}(\mathbf{r},t) = \int \rho\sigma d\sigma$. To apply the statistical theory, one must first specify the constraints attached to the SW equations. The mass $M = \int h d\mathbf{r}$, circulation $\Gamma = \int \overline{q} h d\mathbf{r}$ and energy $E = \int h \frac{\mathbf{u}^2}{2} d\mathbf{r} + \frac{1}{2}\int g h^2 d\mathbf{r}$ will be called *robust constraints* because they can be expressed in terms of the coarse-grained field. These integrals can be calculated at any time from the coarse-grained field and they are conserved by the (macroscopic) dynamics. By contrast, the Casimir invariants $I_f = \int \overline{f(q)} h d\mathbf{r}$, or equivalently the fine-grained moments of the vorticity $\Gamma^{f.g.}_{n>1} = \int \overline{q^n} h d\mathbf{r} = \int \rho\sigma^n d\sigma h d\mathbf{r}$, will be called *fragile constraints* because they must be expressed in terms of the fine-grained PV. Indeed, the moments of the coarse-grained PV $\Gamma^{c.g}_{n>1} = \int \overline{q}^n h d\mathbf{r}$ are *not* conserved since $\overline{q^n} \neq \overline{q}^n$ (part of the coarse-grained moments goes into fine-grained fluctuations). Therefore, the moments $\Gamma^{f.g.}_{n>1}$ must be calculated from the fine-grained field $q(\mathbf{r},t)$ or from the initial conditions, i.e. before the PV vorticity has mixed. Since we often do not know the initial conditions nor the fine-grained field, the Casimir invariants often appear as "hidden constraints" (Chavanis , 2006).

The statistical theory of Miller and Robert-Sommeria-Chavanis for the 2D Euler equations and the SW equations is based on several assumptions: (i) it is assumed that the flow is freely evolving without small-scale forcing and dissipation. (ii) it is assumed that we know the initial conditions (or equivalently the value of all the Casimirs) in detail. (iii) it is assumed that mixing is efficient and that the evolution is ergodic so that the system will reach at equilibrium the most probable (most mixed) state. Within these assumptions [1], the statistical equilibrium state of the SW system is obtained by maximizing

[1] Some attempts have been proposed to go beyond the assumptions (ii) and (iii) of the statistical theory. For example, Chavanis & Sommeria (1996) consider a *strong mixing limit* in which only the first moments of the vorticity are relevant instead of the whole set of Casimirs. On the other hand, Chavanis & Sommeria (1998) introduce the concept of *maximum entropy bubbles* (or restricted equilibrium states) in order to account for situations where the evolution of the flow is not ergodic in the whole available domain but only in subdomains. A 2D turbulent flow is therefore viewed as an ensemble of isolated vortices which can be seen as "maximum entropy bubbles" separated by un-mixed flow. In 2D decaying turbulence, these isolated vortices result from previous mergings and they are expected to correspond to statistical equilibrium states (Laval et al , 2001). In fact, because of *incomplete relaxation*, they may just well be particular stable stationary solutions of the 2D Euler equation that are incompletely mixed.

the mixing entropy

$$S[\rho] = -\int \rho \ln \rho h d\mathbf{r} d\sigma, \tag{15}$$

while conserving the energy, the circulation, the mass and *all the Casimirs*. We write the variational principle in the form

$$\delta S - \beta \delta E - \alpha \delta \Gamma - \gamma \delta M - \sum_{n>1} \alpha_n \delta \Gamma_n^{f.g.} - \int \zeta(\mathbf{r}) \delta \left(\int \rho d\sigma \right) h d\mathbf{r} = 0. \tag{16}$$

In the present point of view, all the constraints are treated on the same footing. In particular, the moments $\Gamma_n^{f.g.}$ are treated microcanonically and we must ultimately relate the Lagrange multipliers α_n to the constraints $\Gamma_n^{f.g.}$.

3.2 Prior distribution and relative mixing entropy

In the approach of Chavanis & Sommeria (2002), it is assumed that the system is strictly described by the SW equations so that the conservation of all the Casimirs has to be taken into account. However, in geophysical situations, the flows are forced and dissipated at small scales (due to convection in the jovian atmosphere) so that the conservation of the Casimirs is destroyed. Ellis et al (2002) have proposed to treat these situations by fixing the conjugate variables $\alpha_{n>1}$ instead of the fragile moments $\Gamma_{n>1}^{f.g.}$. If we view the PV levels as species of particles, this is similar to fixing the chemical potentials instead of the total number of particles in each species. Therefore, the idea is to treat the fragile constraints *canonically*, whereas the robust constraints (E, Γ, M) are still treated *microcanonically*. This point of view has been further developed in Chavanis (2005a) and Chavanis (2005b) in the QG model and this approach is extended here to the SW system. The relevant thermodynamical potential is obtained from the mixing entropy (15) by using a Legendre transform with respect to the fragile constraints:

$$S_\chi = S - \sum_{n>1} \alpha_n \Gamma_n^{f.g.}. \tag{17}$$

Explicating the fine-grained moments, we obtain

$$S_\chi = -\int \rho \left[\ln \rho + \sum_{n>1} \alpha_n \sigma^n \right] h d\mathbf{r} d\sigma. \tag{18}$$

Introducing the function

$$\chi(\sigma) \equiv \exp\Big\{-\sum_{n>1} \alpha_n \sigma^n\Big\}. \tag{19}$$

we get

$$S_\chi[\rho] = -\int \rho \ln\Big[\frac{\rho}{\chi(\sigma)}\Big] h d\mathbf{r} d\sigma, \tag{20}$$

which has the form of a relative entropy. The function $\chi(\sigma)$ is interpreted as a prior vorticity distribution. It is a global distribution of PV fixed by the small-scale forcing. We shall assume that this function is *imposed* by the small-scale forcing so it must be regarded as *given*.

Assuming ergodicity, the statistical equilibrium state is now obtained by maximizing the relative entropy S_χ at fixed energy E, circulation Γ and mass M (robust constraints). The conservation of the Casimirs has been replaced by the specification of the prior $\chi(\sigma)$. From that point, we can repeat the calculations of Chavanis & Sommeria (2002) with almost no modification. The only difference is that we regard the α_n as given. Writing $\delta S_\chi - \beta\delta E - \alpha\delta\Gamma - \gamma\delta M = 0$, and accounting for the normalization condition $\int \rho d\sigma = 1$, we find (i) that the statistical equilibrium state is a stationary solution of the SW equation (ii) that the detailed distribution of PV levels is given by the Gibbs state

$$\rho(\mathbf{r}, \sigma) = \frac{1}{Z}\chi(\sigma) e^{-(\beta\psi+\alpha)\sigma}, \tag{21}$$

where the partition function

$$Z = \int \chi(\sigma) e^{-(\beta\psi+\alpha)\sigma} d\sigma, \tag{22}$$

is determined by the local normalization condition. The distribution of the fluctuations of PV vorticity is the product of a universal Boltzmann factor by a non-universal function $\chi(\sigma)$ fixed by the forcing. This is the same formula as in Chavanis & Sommeria (2002) except that in the present formalism $\chi(\sigma)$ must be regarded as given *a priori* while in Chavanis & Sommeria (2002) it was a function of the Lagrange multipliers α_n that had to be related *a posteriori* to the fine-grained moments imposed by the initial conditions. The equilibrium coarse-grained PV is

$$\overline{q} = \frac{\int \chi(\sigma)\sigma e^{-(\beta\psi+\alpha)\sigma} d\sigma}{\int \chi(\sigma) e^{-(\beta\psi+\alpha)\sigma} d\sigma} = -\frac{1}{\beta}\frac{d\ln Z}{d\psi} = F(\beta\psi+\alpha) = f(\psi). \tag{23}$$

The coarse-grained vorticity (23) can be viewed as a sort of *superstatistics* (Chavanis , 2006) as it is expressed as a superposition of Boltzmann factors (on the fine-grained scale) weighted by a non-universal function $\chi(\sigma)$. We note that the $\overline{q}-\psi$ relationship predicted by the statistical theory can take a wide diversity of forms (usually non-Boltzmannian) depending on the prior $\chi(\sigma)$. The function F is entirely specified by the prior PV distribution according to

$$F(\Phi) = -(\ln \hat{\chi})'(\Phi), \qquad \text{with} \quad \hat{\chi}(\Phi) = \int_{-\infty}^{+\infty} \chi(\sigma) e^{-\sigma\Phi} d\sigma. \tag{24}$$

Differentiating Eq. (23) with respect to ψ, we find that

$$\overline{q}'(\psi) = -\beta q_2, \qquad q_2 = \overline{q^2} - \overline{q}^2 \geq 0. \tag{25}$$

This relation relates the slope of the $\overline{q} = f(\psi)$ relationship to the local centered variance $q_2(\psi)$ of the PV distribution. This can also be written $F'(\Phi) = -q_2(\Phi) \leq 0$ so that F is a decreasing function. Therefore, the statistical theory predicts that the coarse-grained field is a *stationary solution* of the SW equation and that the $\overline{q}-\psi$ relationship is a *monotonic* function which is increasing at negative temperatures $\beta < 0$ and decreasing at positive temperatures $\beta > 0$. We note that, according to Eqs. (12) and (23), the Bernouilli function is given by

$$B = \frac{1}{\beta} \ln Z, \tag{26}$$

and it plays the role of a free energy in the statistical theory (if we interpret Z as a partition function).

We also note that the most probable PV field $\langle\sigma\rangle(\mathbf{r})$ of the Gibbs distribution (21) is given by (Leprovost et al , 2005):

$$\langle\sigma\rangle = [(\ln \chi)']^{-1}(\beta\psi + \alpha), \tag{27}$$

provided that $(\ln \chi)''(\langle\sigma\rangle) < 0$. This is also a stationary solution of the SW system which usually differs from the average value $\overline{q}(\mathbf{r})$ of the Gibbs distribution (21). They coincide only when

$$-(\ln \hat{\chi})'(\Phi) = [(\ln \chi)']^{-1}(\Phi), \tag{28}$$

which is the case when $\chi(\sigma)$ is gaussian.

We first show that a PV field which extremizes a functional of the form

$$H[q] = -\int C(q) h d\mathbf{r}, \tag{29}$$

where C is a convex function, at fixed energy E, mass M and circulation Γ, is a steady state of the SW equations. We write the variational principle as

$$\delta H - \beta\delta E - \alpha\delta\Gamma - \gamma\delta M = 0. \tag{30}$$

Using the results of Chavanis & Sommeria (2002), we have

$$\delta E = \int B\delta h d\mathbf{r} + \int \psi\delta(qh) d\mathbf{r} - \int \phi\delta(\nabla\cdot\mathbf{u}) d\mathbf{r}, \tag{31}$$

$$\delta\Gamma = \int \delta(qh) d\mathbf{r}, \tag{32}$$

$$\delta H = -\int C(q)\delta h d\mathbf{r} - \int C'(q)\delta(qh) d\mathbf{r} + \int C'(q) q\delta h d\mathbf{r}, \tag{33}$$

where we have taken h, qh and $\nabla\cdot\mathbf{u}$ as independent variables. The variations on $\nabla\cdot\mathbf{u}$ yield $\phi = 0$. The variations on qh give

$$C'(q) = -\beta\psi - \alpha, \tag{34}$$

so that $q = f(\psi)$. The variations on h give

$$qC'(q) - C(q) = \beta B + \gamma, \tag{35}$$

so that $B = B(\psi)$. Taking the derivative of Eq. (35) with respect to ψ, we find that

$$qC''(q)\frac{dq}{d\psi} = \beta\frac{dB}{d\psi}. \tag{36}$$

According to Eq. (34), we also have

$$\frac{dq}{d\psi} = -\frac{\beta}{C''(q)}, \tag{37}$$

so that $q(\psi)$ is a monotonic function increasing at negative temperatures and decreasing at positive temperatures. Furthermore, we find that Eq. (36) is equivalent to $q = -dB/d\psi$. Therefore, the optimization problem (30) determines stationary solutions of the SW system. Since H, E, Γ and M are conserved by the SW equations, we can argue, as for the 2D Euler equation (Ellis et al , 2002), that a *maximum* of H at fixed E, Γ and M (if it exists) will be nonlinearly dynamically stable with respect to the SW system. In this dynamical context, H is referred to as a Casimir functional and $E - H$ as an energy-Casimir functional.

Note that the optimization problem (30) can also be justified by a selective decay principle (for $-H$) if we interpret the PV vorticity as the *coarse-grained* PV. Indeed, $-H[\overline{q}]$ calculated with the coarse-grained PV decreases (fragile constraint) while $E[\overline{q}]$, $\Gamma[\overline{q}]$ and $M[\overline{q}]$ are approximately conserved (robust constraints). This *selective decay principle* can explain physically *how* $-H[\overline{q}]$ can possibly reach a minimum value while $-H[q]$ is exactly conserved by the SW equations. In this coarse-grained context, $H[\overline{q}]$ is referred to as a generalized H-function (Tremaine et al , 1986).

Finally, we note that the equilibrium state (23) predicted by the statistical theory extremizes a certain H-function at fixed E, Γ and M. This functional

$$S[\overline{q}] = -\int C(\overline{q}) h d\mathbf{r}, \tag{38}$$

corresponding to the statistical equilibrium state, will be called a generalized entropy in $\overline{q}$-space (Chavanis , 2006). It is completely determined by the prior PV distribution. It should not be confused with the mixing entropy (15) which is a functional of ρ. Coming back to Eq. (34) and recalling that C is convex (so that this relation can be inversed), we find that Eqs. (34) and (23) coincide provided that $C'(\overline{q}) = -F^{-1}(\overline{q})$. Therefore, the prior PV distribution $\chi(\sigma)$ determines $F(x)$ which itself determines $C(\overline{q})$ according to

$$C(\overline{q}) = -\int^{\overline{q}} F^{-1}(x) dx. \tag{39}$$

In other words, to obtain $C(\overline{q})$, we need to inverse Eq. (23) and integrate the resulting expression. Some examples are collected in Chavanis (2003). Combining the previous relations, we find that the generalized entropy is determined by the prior according to (Chavanis , 2006)

$$C(\overline{q}) = -\int^{\overline{q}} [(\ln \hat{\chi})']^{-1}(-x) dx. \tag{40}$$

Finally, comparing Eqs. (37) and (25) we get the relation

$$q_2 = \frac{1}{C''(\overline{q})}, \tag{41}$$

which is exact at statistical equilibrium.

4 Relaxation equations

4.1 Maximum Entropy Production Principle

In the case of freely evolving flows, Chavanis & Sommeria (2002) have proposed a thermodynamical parametrization of the SW equations (on the coarse-grained scale) in the form of relaxation equations that conserve all the constraints of the SW dynamics (including the Casimirs) and increase the mixing entropy. In the case of flows that are forced at small-scales, the Casimirs are replaced by the specification of a prior vorticity distribution or, as we have seen, by a generalized entropy. In this context, we can propose a thermodynamical parametrization of the SW equations in the form of relaxation equations that conserve only the robust constraints (mass, energy and circulation) and increase the generalized entropy $S[\overline{q}]$ fixed by the prior.

We first decompose the vorticity ω and velocity $\mathbf{u}$ into a mean and fluctuating part, namely $\omega = \overline{\omega} + \tilde{\omega}$, $\mathbf{u} = \overline{\mathbf{u}} + \tilde{\mathbf{u}}$, keeping h smooth. Taking the local average of the shallow water equations (1)(3), we get

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\overline{\mathbf{u}}) = 0, \tag{42}$$

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + (\overline{\boldsymbol{\omega}} + 2\boldsymbol{\Omega}) \times \overline{\mathbf{u}} = -\nabla B - \mathbf{e}_z \times \mathbf{J}_\omega, \tag{43}$$

where the current $\mathbf{J}_\omega = \overline{\tilde{\omega}\tilde{\mathbf{u}}}$ represents the correlations of the fine-grained fluctuations. We deduce an equation for the evolution of the potential vorticity (5), taking the curl of Eq. (43) and using Eq. (42):

$$\frac{\partial}{\partial t}(h\overline{q}) + \nabla \cdot (h\overline{q}\,\overline{\mathbf{u}}) = -\nabla \cdot \mathbf{J}_\omega. \tag{44}$$

This equation can be viewed as a local conservation law for the circulation $\Gamma = \int \overline{q} h d\mathbf{r}$. It shows also that $\mathbf{J}_\omega$ represents the current of coarse-grained vorticity due to mixing. We shall determine the unknown current $\mathbf{J}_\omega$ by the thermodynamic principle of Maximum Entropy Production (MEP), using the generalized entropy (38). The Maximum Entropy Production (MEP) principle consists in choosing the current $\mathbf{J}_\omega$ which maximizes the rate of entropy production $\dot{S}$ respecting the constraints $\dot{E} = 0$, and $J_\omega^2 \leq C(\mathbf{r}, t)$. The last constraint expresses a bound (unknown) on the value of the diffusion current. Convexity arguments justify that this bound is always reached so that the inequality can be replaced by an equality. We write the variational problem as

$$\delta\dot{S} - \beta(t)\delta\dot{E} - \int D^{-1}\delta\left(\frac{J_\omega^2}{2}\right)d\mathbf{r} = 0, \tag{45}$$

where $\beta(t)$ is a Lagrange multiplier accounting for the conservation of energy and D^{-1} is a Lagrange multiplier associated with the constraint $J_\omega^2 = C(\mathbf{r}, t)$. The conservations of mass and circulation are automatically satisfied by the form of the relaxation equations (42) and (43). Noting that

$$\dot{E} = \int \mathbf{J}_\omega \cdot \overline{\mathbf{u}}_\perp h d\mathbf{r}, \tag{46}$$

$$\dot{S} = -\int C''(\overline{q})\mathbf{J}_\omega \cdot \nabla\overline{q} d\mathbf{r}, \tag{47}$$

and performing the variations on $\mathbf{J}_\omega$ in Eq. (45), we obtain an optimal current

$$\mathbf{J}_\omega = -D\left[\nabla\overline{q} + \frac{\beta(t)}{C''(\overline{q})}h\overline{\mathbf{u}}_\perp\right]. \tag{48}$$

Thus, in the presence of a prior PV distribution, we obtain a parametrization of the SW equations of the form

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \tag{49}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \overline{q}h\mathbf{e}_z \times \mathbf{u} = -\nabla\left(\frac{\mathbf{u}^2}{2} + gh\right) + D\left[\mathbf{e}_z \times \nabla\overline{q} - \frac{\beta(t)}{C''(\overline{q})}h\mathbf{u}\right], \tag{50}$$

$$\overline{q} = \frac{(\nabla \times \mathbf{u}) \cdot \mathbf{e}_z + 2\Omega}{h}, \quad \beta(t) = -\frac{\int Dh\mathbf{u}_\perp \cdot \nabla\overline{q} d\mathbf{r}}{\int D\frac{\mathbf{u}^2 h^2}{C''(\overline{q})}d\mathbf{r}}, \tag{51}$$

$$\mathbf{n} \cdot \nabla\overline{q} = -\frac{\beta(t)}{C''(\overline{q})}h\ \mathbf{n} \cdot \mathbf{u}_\perp \qquad \text{(on each boundary)}, \tag{52}$$

$$\mathbf{n} \cdot \mathbf{u} = 0 \qquad \text{(on each boundary)}, \tag{53}$$

where $\mathbf{n}$ is a unit vector normal to the boundary and we have omitted the overbar on $\mathbf{u}$. The relaxation equation for the coarse-grained vorticity is given by

$$\frac{\partial}{\partial t}(h\overline{q}) + \nabla\cdot(h\overline{q}\mathbf{u}) = \nabla\cdot\left\{D\left[\nabla\overline{q} + \frac{\beta(t)}{C''(\overline{q})}h\overline{\mathbf{u}}_{\perp}\right]\right\}. \tag{54}$$

These equations can also be directly obtained from the parametrization of Chavanis & Sommeria (2002) by replacing q_2 in their parametrization by $1/C''(q)$. Therefore, the identity (41) can be viewed as a *closure relationship* in the present context. This relation is valid at equilibrium but the present approach suggests that it is also valid out-of-equilibrium when there is a prior distribution of PV. In fact, we can obtain this relation by assuming that, out-of-equilibrium, the PV distribution $\rho(\mathbf{r},\sigma,t)$ maximizes the relative entropy (20) at fixed PV $\overline{q}(\mathbf{r},t)$ and normalization (Appendix C of Chavanis (2005a)).

The entropy production (47) can be written

$$\dot{S} = -\int C''(\overline{q})\mathbf{J}_{\omega}\cdot\left[\nabla\overline{q} + \frac{\beta(t)}{C''(\overline{q})}h\mathbf{u}_{\perp}\right]d\mathbf{r} + \beta(t)\int \mathbf{J}_{\omega}\cdot\mathbf{u}_{\perp}hd\mathbf{r}. \tag{55}$$

Using the conservation of energy $\dot{E}=0$ with Eq. (46), the second integral is seen to vanish. Substituting Eq. (48) in the first integral, we finally obtain

$$\dot{S} = \int C''(\overline{q})\frac{\mathbf{J}_{\omega}^2}{D}d\mathbf{r} \geq 0, \tag{56}$$

which is positive provided that $D \geq 0$. A stationary solution of the relaxation equations (49)-(53) satisfies $\dot{S}=0$ yielding $\mathbf{J}_{\omega}=\mathbf{0}$, i.e.

$$\nabla\overline{q} + \frac{\beta}{C''(\overline{q})}\nabla\psi = \mathbf{0}. \tag{57}$$

After integration, we obtain

$$C'(q) = -\beta\psi - \alpha. \tag{58}$$

Therefore, the generalized entropy increases until the statistical equilibrium state (23)-(34), fixed by the prior $\chi(\sigma)$, is reached with $\beta = \lim_{t\to+\infty}\beta(t)$. Alternatively, these equations can be used as a numerical algorithm to compute arbitrary stationary solutions of the SW system specified by the convex function C (see Sec. 3.3).

The case of ordinary 2D incompressible turbulence is recovered in the limit $h \to 1$, $q \to \omega$ and $\mathbf{u} = -\mathbf{e}_z \times \nabla\psi$. The relaxation equation for the coarse-grained vorticity is given by

$$\frac{\partial \overline{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \overline{\omega} = \nabla \cdot \left\{ D \left[\nabla \overline{\omega} + \frac{\beta(t)}{C''(\overline{\omega})} \nabla \psi \right] \right\}, \tag{59}$$

with

$$\beta(t) = -\frac{\int D \nabla \overline{\omega} \cdot \nabla \psi d\mathbf{r}}{\int D \frac{(\nabla\psi)^2}{C''(\overline{\omega})} d\mathbf{r}}. \tag{60}$$

This returns the equations introduced by Chavanis (2003) in 2D turbulence in the case where the system is described by a prior vorticity distribution. They can be viewed as nonlinear mean-field Fokker-Planck equations. They are the forced-case counterpart of the relaxation equations introduced by Robert & Sommeria (1992) for freely evolving flows that conserve all the Casimir constraints.

The relaxation equation (50) for the velocity field can be written

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho} \nabla p + D \left[\Delta \mathbf{u} - \frac{\beta(t)}{C''(\overline{\omega})} \mathbf{u} \right], \tag{61}$$

where we have used the well-known identity of vector analysis $\Delta \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$ which reduces for a two-dimensional incompressible flow to $\Delta \mathbf{u} = \mathbf{e}_z \times \nabla \overline{\omega}$. Eq. (61) is valid even if D is space dependant unlike with a usual viscosity term. In previous publications this equation was given only in its vorticity form (59) and the equivalence with Eq. (61) is not obvious at first sights when D is space dependent. At equilibrium, we have from Eq. (61) the identity

$$\Delta \mathbf{u} = \frac{\beta}{C''(\overline{\omega})} \mathbf{u}, \tag{62}$$

which can be deduced directly from the stationary state (34). Indeed, for a stationary solution $\overline{\omega} = f(\psi)$, the previous identity $\Delta \mathbf{u} = \mathbf{e}_z \times \nabla \overline{\omega}$ becomes $\Delta \mathbf{u} = -f'(\psi)\mathbf{u}$ which is equivalent to Eq. (62) for a steady state thanks to Eq. (37). Finally, the previous equations can be extended to the quasi-geostrophic

(QG) limit if we replace ω by the PV q related to the stream-function by

$$q = -\Delta\psi + \frac{\psi}{L_R^2}, \tag{63}$$

where L_R is the Rossby radius.

4.3 Explicit examples

In the present formalism, one has to specify (i) a prior PV distribution $\chi(\sigma)$ which encodes small-scale forcing and non-ideal effects (ii) the robust constraints E, Γ and M which can be determined at any time from the coarse-grained flow. From the prior PV distribution, we can determine a generalized entropy $C(\overline{q})$ by using the procedure exposed in Sec. 3.3 (see in particular Eq. (40)). Then, we can use this entropy in the parametrization (49)-(53) to obtain the dynamical evolution of the coarse-grained flow towards statistical equilibrium. The difficulty is now to find the good prior. This depends from case to case as it is related to the properties of forcing, but the idea is that several forms of prior (or corresponding entropies) give similar results so that they can be regrouped in *classes of equivalence* (Chavanis , 2003). Thus, for a given situation, one has to find the relevant class of equivalence. In general, one has to proceed by tryings and errors. We specify a prior, compute the corresponding flow and see whether it agrees with the information that we have on the system. If we find a "good prior" for a system with some given initial conditions, then we can expect that it will remain valid for this system when we change the initial conditions (or, equivalently, the value of the robust constraints E, Γ and M). Note that specifying the prior is not the end of the story but only the starting point. Indeed, there can be different types of solutions for a given prior PV distribution depending on the values of the control parameters. Thus, we have to study the bifurcation diagram as a function of these parameters E, Γ and M for a given prior $\chi(\sigma)$ or generalized entropy $C(\overline{q})$. This is a rich and non trivial problem.

Let us give some examples to illustrate our approach. In the case of Jovian flows, Ellis et al (2002) have proposed to adopt a prior $\chi(\sigma)$ corresponding to a de-centered Gamma distribution. This leads to a generalized entropy of the form (Chavanis , 2006)

$$C(\overline{q}) = \frac{1}{\epsilon}\left[\overline{q} - \frac{1}{\epsilon}\ln(1+\epsilon\overline{q})\right], \tag{64}$$

where 2ϵ is the skewness of the PV distribution. We have $C'(\overline{q}) = \overline{q}/(1+\epsilon\overline{q})$ and $C''(\overline{q}) = 1/(1+\epsilon\overline{q})^2$ so that the statistical equilibrium state is specified by

$$\overline{q} = -\frac{\beta\psi + \alpha}{1 + \epsilon(\beta\psi + \alpha)}. \tag{65}$$

In the limit $\epsilon \to 0$, the generalized entropy (64) becomes minus the enstrophy $S[\overline{q}] = -\frac{1}{2}\int \overline{q}^2 d\mathbf{r}$ and the $\overline{q} = f(\psi)$ relationship is linear. The parametrization that we propose in that case for the vorticity equation is

$$\frac{\partial}{\partial t}(h\overline{q}) + \nabla \cdot (h\overline{q}\mathbf{u}) = \nabla \cdot \left\{ D\left[\nabla\overline{q} + \beta(t)(1 + \epsilon\overline{q})^2 h\overline{\mathbf{u}}_\perp \right] \right\}. \tag{66}$$

On the other hand, in order to describe jovian flows, Sommeria et al (1991) have considered a case of the statistical theory where the PV distribution is restricted to two-levels. In their approach, the flow is assumed to be freely evolving and the dynamics corresponds to the inviscid mixing of patches with PV σ_0 and σ_1. As discussed in Chavanis (2005a), we can also consider the case of a flow that is permanently forced at small-scales (due to convection) so that a prior PV distribution is established with two intense peaks at σ_0 and σ_1. In the two-levels case, theses two interpretations lead to the same results but the second one can probably be extended more easily to the more realistic situation where the peaks have a finite width $\Delta\sigma$. If we adopt a prior of the form $\chi(\sigma) = \delta(\sigma - \sigma_1) + \chi\delta(\sigma - \sigma_0)$, we find that the generalized entropy is

$$C(\overline{q}) = \frac{1}{\sigma_1 - \sigma_0}\left[(\overline{q} - \sigma_0)\ln(\overline{q} - \sigma_0) + (\sigma_1 - \overline{q})\ln(\sigma_1 - \overline{q}) \right]. \tag{67}$$

We have $C'(\overline{q}) = \frac{1}{\sigma_1 - \sigma_0}\ln(\frac{\overline{q} - \sigma_0}{\sigma_1 - \overline{q}})$ and $C''(\overline{q}) = 1/[(\overline{q} - \sigma_0)(\sigma_1 - \overline{q})]$ so that the statistical equilibrium state is

$$\overline{q} = \sigma_0 + \frac{\sigma_1 - \sigma_0}{1 + \chi e^{(\sigma_1 - \sigma_0)(\beta\psi + \alpha)}}, \tag{68}$$

similar to the Fermi-Dirac distribution. The parametrization that we propose in that case for the vorticity equation is

$$\frac{\partial}{\partial t}(h\overline{q}) + \nabla \cdot (h\overline{q}\mathbf{u}) = \nabla \cdot \left\{ D\left[\nabla\overline{q} + \beta(t)(\overline{q} - \sigma_0)(\sigma_1 - \overline{q}) h\overline{\mathbf{u}}_\perp \right] \right\}, \tag{69}$$

and it coincides with the parametrization of Chavanis & Sommeria (2002) in the two-levels case of the statistical theory. Therefore, the approach based on priors is not in radical opposition with the usual statistical theory but it allows for convenient extensions in more general cases.

Explicit determinations of the statistical equilibrium state specified by Eqs. (65) and (68) have been obtained in the QG limit of the statistical theory. In the two approaches, a vortex solution has been found at the latitude of Jupiter's great red spot where the underlying topography is extremum (or the shear is equal to zero). In the case of the Fermi-Dirac distribution (68), Sommeria et al (1991) consider the limit of small Rossby radius and find that the vortex has an annular jet structure which is consistent with the morphology of Jupiter's great red spot. This study has been further developed in Bouchet & Sommeria (2002) and it has been extended by Bouchet et al (2006) to the case of the SW system. In these studies, the structure of Jupiter's great red spot can be seen as the co-existence of two thermodynamical phases in contact separated by a sort of "domain wall" (Chavanis , 2005a) in the langage of phase ordering kinetics (like in the Cahn-Hilliard theory). This annular jet structure is not obtained in the approach of Ellis et al (2002). This implies that the prior PV distribution relevant to the case of JGRS is probably closer to two intense peaks rather than to a decentered Gamma distribution. These two distributions belong to different classes of equivalence since they lead to structurally different solutions. Probably, the prior distribution could be improved to give a better description of jovian vortices, but a distribution with two intense peaks already gives a fair description.

5 Conclusion

In this paper, we have extended the "ordinary" statistical theory of the SW system (Chavanis & Sommeria , 2002) to account for the existence of a prior vorticity distribution. This approach can be justified when the system is forced at small-scales so that a permanent PV distribution is imposed *canonically*. In that case, the forcing acts like a sort of reservoir. The attractive nature of this theory is its *practical interest*: in the standard theory, one works with the Boltzmann entropy in ρ-space and deals with a very large (possibly infinite) number of Casimir constraints which are often not known or not rigorously conserved. In the other approach, one conserves only the robust constraints (E, Γ, M) and work with a generalized entropic functional in $\overline{q}$-space fixed by the prior. Therefore, in the first approach, we have to solve N coupled relaxation equations (one for each level) while in the second approach, we just have to solve one relaxation equation. Whether this approach is really physically relevant remains to be established. In any case, the relaxation equations (49)-(53) can be used as numerical algorithms to construct a large class of stationary solutions of the SW equations, which is certainly an interest of our formalism.

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