Turbulent Cascade of Circulations

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Abstract

The circulation around any closed loop is a Lagrangian invariant for classical, smooth solutions of the incompressible Euler equations in any number of space dimensions. However, singular solutions relevant to turbulent flows need not preserve the classical integrals of motion. Here we generalize the Kelvin theorem on conservation of circulations to distributional solutions of Euler and give necessary conditions for the anomalous dissipation of circulations. We discuss the important role of Kelvin's theorem in turbulent vortex-stretching dynamics and conjecture a version of the theorem which may apply to suitable singular solutions.

Résumé

Cascade turbulente des circulations La circulation autour d'une boucle fermée est un invariant de Lagrange pour des solutions classiques des équations d'Euler incompressibles dans un espace dimension quelconque. Cependant, les solutions singulières qui s'appliquent à des écoulements turbulents ne conservent pas forcément les constantes du mouvement classiques. Dans cette contribution, nous généralisons le théorème de Kelvin sur la conservation des circulations aux solutions singulières d'Euler et donnons les conditions nécessaires pour la dissipation anormale des circulations. Nous discutons le rôle important du théorème de Kelvin dans la dynamique turbulente qui déforme les tourbillons, et proposons une version du théorème qui pourrait s'appliquer aux solutions singulières.

Key words: turbulence, circulation, Euler equations, Kelvin theorem

1 Introduction

In a monumental paper, Helmholtz (1858) formulated the fundamental laws of vortex motion for incompressible inviscid fluids. These include the statements,

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in three space dimensions, that vortex lines are material lines and that the flux within any vortex tube is a Lagrangian invariant. Lord Kelvin (1869) gave an elegant alternative formulation of these laws in terms of the conservation of circulation, for any closed loop advected by an ideal fluid. This theorem is equally valid in any space dimension. However, it is only rigorously proved for sufficiently smooth solutions. As was pointed out by Onsager (1949), the classical conservation laws need not be valid for singular solutions of Euler equations. In particular, breakdown of the energy conservation law can account for the anomalous dissipation of energy observed in turbulent fluids at high Reynolds numbers. From a physical point of view, this breakdown of energy conservation corresponds to the turbulent energy cascade and a flux of energy to arbitrarily small scales. See also Eyink (1994),Constantin et al. (1994), Duchon & Robert (2000).

These considerations make it very natural to inquire whether Kelvin's theorem will remain valid for singular solutions of the Euler equations. This question asumes some importance since the conservation of circulations was argued by Taylor (1938) to play a key role in the enhanced production of dissipation in turbulent fluids, by the process of vortex line-stretching. Despite its plausibility, the validity of Taylor's argument is far from clear. It is not obvious *a priori* why there should not be anomalous dissipation of the circulation invariants, corresponding to a turbulent "flux of circulations" from large to small scales.

In this paper, we examine these questions and establish a few relevant rigorous results. In section 2 we briefly review the classical Kelvin-Helmholtz theorem and its role in turbulence dynamics. In section 3 we prove a theorem on conservation of circulations for singular solutions of incompressible Euler equations, analogous to that of Onsager (1949) for conservation of energy. In section 4 we discuss difficulties in formulating the Kelvin theorem for singular solutions, due to the breakdown in uniqueness of Lagrangian trajectories, and conjecture a statistical version of circulation-conservation which may apply.

2 The Classical Kelvin Theorem

The velocity field $\mathbf{u}(\mathbf{x}, t)$ solving the incompressible Navier-Stokes equation

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu \triangle \mathbf{u}, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \tag{1}$$

with $\mathbf{x} \in \Lambda \subset \mathbf{R}^d$, for any integer $d \geq 2$, satisfies the Kelvin-Helmholtz theorem in the following sense: For any closed, rectifiable loop $C \subset \Lambda$ at an initial time t_0, the circulation $\Gamma(C,t) = \oint_{C(t)} \mathbf{u}(t)\cdot d\mathbf{x}$ satisfies

$$\frac{d}{dt}\Gamma(C,t) = \nu \oint_{C(t)} \triangle\mathbf{u}(t)\cdot d\mathbf{x}, \tag{2}$$

where $C(t)$ is the loop advected by the fluid velocity, at time t. E.g., see Saffman (1992), section §1.6, for the standard derivation. It is worth observing that the Kelvin theorem for all loops C is formally equivalent to the Navier-Stokes equation (1). Indeed, if $\mathbf{u}(\mathbf{x},t)$ is a smooth spacetime velocity field, diverence-free at all times t, then equation (2) implies that

$$\oint_C [D_t\mathbf{u}(t) - \nu \triangle \mathbf{u}(t)] \cdot d\mathbf{x} = 0 \tag{3}$$

for all loops C at every time t. Here $D_t\mathbf{u} = \partial_t\mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$ is the Lagrangian time-derivative and the equation (3) is derived by applying (2) to the pre-image of the loop C at initial time t_0. By Stokes theorem, equation (3) can hold for all loops $C \subset \Lambda$ if and only if there exists a pressure-field $p(\mathbf{x},t)$ such that the Navier-Stokes equation (1) holds locally and also globally, if the domain Λ is simply connected.

In the inviscid limit $\nu \to 0$, the circulation is formally conserved for any initial loop C. The fluid equations in this limit, the incompressible Euler equations, are the equations of motion of a classical Hamiltonian system. They can be derived by the Hamilton-Maupertis principle from the action functional

$$S[\mathbf{x}] = \frac{1}{2}\int_{t_0}^{t_f} dt \int_\Lambda d\mathbf{a}\ |\dot{\mathbf{x}}(\mathbf{a},t)|^2 \tag{4}$$

with the pressure field $p(\mathbf{x},t)$ a Lagrange multiplier to enforce the incompressibility constraint. Here $\mathbf{x}(\mathbf{a},t)$ is the Lagrangian flow map which satisfies $\dot{\mathbf{x}}(\mathbf{a},t) = \mathbf{u}(\mathbf{x}(\mathbf{a},t),t)$ with initial condition $\mathbf{x}(\mathbf{a},t_0) = \mathbf{a}$. See Salmon (1988) for a review. This variational principle yields the fluid equations in a Lagrangian formulation, as $\ddot{\mathbf{x}}(\mathbf{a},t) = -\boldsymbol{\nabla}p(\mathbf{x}(\mathbf{a},t),t)$. The Eulerian formulation (1) (with $\nu = 0$) is obtained by performing variations in the inverse map $\mathbf{a}(\mathbf{x},t)$, or "back-to-labels map", with fixed particle positions $\mathbf{x}$. This Hamiltonian system has an infinite-dimensional gauge symmetry group consisting of all volume-preserving diffeomorphisms of Λ, which corresponds to all smooth choices of initial fluid particle labels. In this framework, the conservation of the circulations for all closed loops C emerges as a consequence of Noether's theorem for the particle-relabelling symmetry. See Salmon (1988), Section 4.

The Kelvin theorem has many profound consequences for the dynamics of incompressible fluids. We just note here a well-known deduction for three-dimensional turbulence by G. I. Taylor (1938). It is reasonable to assume that vortex lines—or, for that matter, any material lines—will tend to elongate under chaotic advection by a turbulent velocity field. Incompressibility requires that the cross-sectional area of a vortex tube formed by such lines will shrink with time. But, in that case, the Kelvin-Helmholtz theorem implies that the vorticity magnitude of the tube must grow. Taylor (1938) observed that this process of vortex line-stretching provides an intrinsic mechanism for amplification of the net viscous dissipation $\nu \int_\Lambda d\mathbf{x}|\boldsymbol{\omega}|^2$, where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$. More recently, regularizations of the Navier-Stokes equation have been proposed as model equations for large-scale turbulence by Holm et al. (1998); Foias et al. (2001), motivated by requiring that a Kelvin circulation theorem be preserved.

3 A Generalized Theorem on Conservation of Circulation

This line of reasoning of Taylor (1938) is quite delicate. It assumes certain properties (material advection of vortex lines and conservation of circulations) that can hold strictly only in the $\nu \to 0$ limit. However, conclusions are drawn about the limiting behavior of the energy dissipation rate, which directly involves the kinematic viscosity ν! Furthermore, as noted by Onsager (1949), the solutions of the Navier-Stokes equation are not expected to remain smooth in the inviscid limit. Thus, the righthand side of equation (2) does not necessarily vanish as $\nu \to 0$. Physically, there may be a dissipative anomaly for the conservation of circulations. If so, then the validity of Taylor's vortex-stretching mechanism for turbulent energy dissipation is open to serious question.

The formulation of turbulent conservation of circulation by a zero-viscosity limit is physically natural, but not the most convenient either for numerical tests or for rigorous mathematical analysis [1]. We shall instead consider directly the singular (or distributional) solutions $\mathbf{u} \in L^2([0,T],\Lambda)$ of the incompressible Euler equations, with $\nu = 0$. Let $\overline{\mathbf{u}}_\ell = G_\ell * \mathbf{u}$ denote the low-pass filtered velocity at length-scale ℓ, where $G_\ell(\mathbf{r}) = \ell^{-d}G(\mathbf{r}/\ell)$ is a smooth filter kernel. Then $\overline{\mathbf{u}}_\ell$ satisfies the following equation (in the sense of distributions in time):

$$\partial_t\overline{\mathbf{u}}_\ell + (\overline{\mathbf{u}}_\ell\boldsymbol{\cdot}\boldsymbol{\nabla})\overline{\mathbf{u}}_\ell = -\boldsymbol{\nabla}\overline{p}_\ell + \mathbf{f}_\ell, \tag{5}$$

[1] For example, the Kelvin theorem in the form of equation (2) is not proved to be valid for the global solutions of the Navier-Stokes equation (1) constructed by Leray (1934). The difficulty here is that the Leray regularization of (1) does not preserve the Kelvin theorem, while alternative regularizations which do, such as that of Foias et al. (2001), in turn modify the energy balance. See Constantin (2003).

where $\overline{p}_\ell$ is the filtered pressure and where $\mathbf{f}_\ell = -\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\tau}_\ell$ is the subgrid force, i.e. minus the divergence of the stress-tensor $\boldsymbol{\tau}_\ell = \overline{(\mathbf{u}\,\mathbf{u})}_\ell - \overline{\mathbf{u}}_\ell\overline{\mathbf{u}}_\ell$. Let us choose a rectifiable closed loop C in space. We define $\overline{C}_\ell(t)$ as the loop C advected by the filtered velocity $\overline{\mathbf{u}}_\ell$. This definition makes sense, since the filtered velocity $\overline{\mathbf{u}}_\ell$ is Lipschitz in space, and corresponding flow maps exist and are unique (DiPerna & Lions (1989)). We define a "large-scale circulation" with initial loop C as the line-integral $\overline{\Gamma}_\ell(C,t) = \oint_{\overline{C}_\ell(t)} \overline{\mathbf{u}}_\ell(t)\cdot d\mathbf{x}$. The same calculation that establishes the Kelvin circulation theorem for smooth solutions of Euler equations gives that

$$\overline{\Gamma}_\ell(C,t) - \overline{\Gamma}_\ell(C,t_0) = \int_{t_0}^{t} d\tau \oint_{\overline{C}_\ell(\tau)} \mathbf{f}_\ell(\tau)\cdot d\mathbf{x}. \tag{6}$$

Thus, the line-integral of $\mathbf{f}_\ell$ on the RHS represents a "flux" to subgrid modes at length-scales $< \ell$ of circulation on the loop $\overline{C}_\ell(\tau)$. This motivates the definition, for any loop C and filter length ℓ,

$$K_\ell(C,t) = -\oint_{C(t)} \mathbf{f}_\ell(t)\cdot d\mathbf{x} \tag{7}$$

so that (in generalized sense) $(d/dt)\overline{\Gamma}_\ell(C,t) = -K_\ell(C,t)$.

We now prove the following:

Theorem: *Let ζ_p be the pth-order scaling exponent of the velocity, in the sense that it is the maximal value such that*

$$\frac{1}{|\Lambda|}\int_{\Lambda} d^d\mathbf{x}|\delta\mathbf{u}(\mathbf{r};\mathbf{x})|^p = O(|\mathbf{r}|^{\zeta_p}),$$

for all $|\mathbf{r}| \le r_0$, where $\delta\mathbf{u}(\mathbf{r};\mathbf{x}) = \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$. Then for any smooth loop $C \subset \Lambda$

$$K_\ell(C) = -\oint_C \mathbf{f}_\ell\cdot d\mathbf{x}$$

satisfies $\lim_{\ell\to 0} K_\ell(C) = 0$ if $\zeta_p > (d-1) + (p/2)$ for any $p \ge 2$.

The special case of this result for $p = \infty$ states that the "circulation flux" will go to zero as $\ell \to 0$ if the smallest velocity Hölder exponent $h_{\min}$ is $> 1/2$. This is an exact analogue of the result of Onsager (1949) for vanishing of energy flux when $h_{\min} > 1/3$. One can see that it is even easier for circulation-conservation to be anomalous than for energy-conservation.

Proof: Our argument is close to that given by Constantin et al. (1994) for the Onsager theorem. The following identity for the subgrid force is easily verified:

$$f_i(\mathbf{x}) = \int d^d\mathbf{r} (\partial_j G)(\mathbf{r})\, \delta u_i(\mathbf{r};\mathbf{x}) \delta u_j(\mathbf{r};\mathbf{x}) - \int d^d\mathbf{r} (\partial_j G)(\mathbf{r})\, \delta u_i(\mathbf{r};\mathbf{x}) \int d^d\mathbf{r}'\, G(\mathbf{r}')\, \delta u_j(\mathbf{r}';\mathbf{x})$$

We omit here all subscripts ℓ for convenience. By this identity,

$$\oint_C \mathbf{f}\cdot d\mathbf{x} = \int d^d\mathbf{r} (\partial_j G)(\mathbf{r}) \left[\oint_C \delta u_i(\mathbf{r}) \delta u_j(\mathbf{r}) dx_i\right] - \int d^d\mathbf{r} (\partial_j G)(\mathbf{r}) \int d^d\mathbf{r}'\, G(\mathbf{r}') \left[\oint_C \delta u_i(\mathbf{r}) \delta u_j(\mathbf{r}') dx_i\right]$$

Thus,

$$\left|\oint_C \mathbf{f}\cdot d\mathbf{x}\right| \leq \int d^d\mathbf{r} |\boldsymbol{\nabla} G(\mathbf{r})| \left[\oint_C |\delta\mathbf{u}(\mathbf{r})|^2 ds\right] + \int d^d\mathbf{r} |\boldsymbol{\nabla} G(\mathbf{r})| \int d^d\mathbf{r}'\, G(\mathbf{r}') \left[\oint_C |\delta\mathbf{u}(\mathbf{r})|\, |\delta\mathbf{u}(\mathbf{r}')| ds\right]$$

where s denotes arclength along the curve C. By normalization $\int d^d\mathbf{r}'\, G(\mathbf{r}') = 1$ and the inequality $|\delta\mathbf{u}(\mathbf{r})|\, |\delta\mathbf{u}(\mathbf{r}')| \leq \frac{1}{2} \left[|\delta\mathbf{u}(\mathbf{r})|^2 + |\delta\mathbf{u}(\mathbf{r}')|^2\right]$, this becomes

$$\left|\oint_C \mathbf{f}\cdot d\mathbf{x}\right| \leq \frac{3}{2} \int d^d\mathbf{r} |\boldsymbol{\nabla} G(\mathbf{r})| \left[\oint_C |\delta\mathbf{u}(\mathbf{r})|^2 ds\right] + \frac{1}{2} \int d^d\mathbf{r} |\boldsymbol{\nabla} G(\mathbf{r})| \int d^d\mathbf{r}'\, G(\mathbf{r}') \left[\oint_C |\delta\mathbf{u}(\mathbf{r}')|^2 ds\right] \quad (8)$$

We now use the Hölder inequality to derive the bound

$$\oint_C |\delta\mathbf{u}(\mathbf{r})|^2 ds \leq [L(C)]^{(p-2)/p} \left(\oint_C |\delta\mathbf{u}(\mathbf{r})|^p ds\right)^{2/p} \quad (9)$$

for any $p \geq 2$, where $L(C)$ is the length of the curve C. The condition on the scaling exponent in the statement of the theorem can rephrased as the condition that $\mathbf{u}$ belong to the Besov space $B^{\sigma_p}_{p,\infty}(\Lambda)$ with $\sigma_p = \zeta_p/p$. (More properly, we should replace σ_p by $\sigma_p - \epsilon$ for any small $\epsilon > 0$.) Standard trace theorems then imply that the restriction of $\mathbf{u}$ to the submanifold C of codimension $d-1$ must satisfy $\mathbf{u}|_C \in B^{\sigma_p-(d-1)/p}_{p,\infty}(C)$. See Triebel (1983), Theorem 2.7.2. Together with inequality (9), this implies that $\oint_C |\delta\mathbf{u}(\mathbf{r})|^2 ds =$

$O\left(|\mathbf{r}|^{2[\zeta_p-(d-1)]/p}\right)$ [2]. If this bound is substituted into estimate (8) for circulation-flux, it yields

$$\left|\oint_C \mathbf{f}\cdot d\mathbf{x}\right| = O\left(\ell^{2[\zeta_p-(d-1)]/p-1}\right).$$

We see that the latter goes to zero as $\ell \to 0$, if $\zeta_p > p/2 + (d-1)$. □

As an application, consider a velocity field that is Lipschitz regular, so that $\zeta_p = p$ for all $p \geq 1$. In that case, it suffices to take $p > 2(d-1)$ in order to show that the circulation is conserved (in the sense that the flux vanishes for $\ell \to 0$.) This result applies to the 2D enstrophy cascade, since it is expected there that $\zeta_p = p$, with only logarithmic corrections, for all $p \geq 2$. See Eyink (2000). Thus, we expect that the Kelvin theorem holds in a strong sense — for individual realizations—in the 2D enstrophy cascade. However, in the 3D energy cascade the conditions of the theorem are not expected to be satisfied.

4 The Role of Kelvin's Theorem for Singular Solutions

Even assuming that the assumptions of our theorem are met, there are additional difficulties in justifying constancy of the circulation invariants. Vanishing of the circulation flux for loops of finite length, as established in our theorem, is not sufficient. In the first place, the material loop $\overline{C}_\ell(t)$ is not expected to remain rectifiable as $\ell \to 0$, but instead to become a fractal curve $C(t)$ with Hausdorff dimension > 1 for any positive time t (Sreenivasan & Meneveau (1986)). Thus, we cannot immediately infer that the RHS of equation (6) vanishes as $\ell \to 0$, nor even make sense of the contour integral in that limit. A possible approach here is to transform the RHS to label-space, as $\oint_C \mathbf{f}_\ell(\overline{\mathbf{x}}_\ell(\tau),\tau)\cdot d\overline{\mathbf{x}}_\ell(\tau)$ where the map satisfies $\dot{\overline{\mathbf{x}}}_\ell(\mathbf{a},\tau) = \overline{\mathbf{u}}_\ell(\overline{\mathbf{x}}_\ell(\mathbf{a},\tau),\tau)$. This can make sense as a Stieltjes integral on the loop C in label-space for Hölder continuous maps (e.g. see Young (1936)),Zähle (1998)).

However, there is a much more serious problem in formulating Kelvin's theorem for singular Euler solutions: it is not clear that material loops exist! Recent work on an idealized turbulence problem—the Kraichnan model of

[2] Only the case $p = \infty$ rigorously follows from standard trace theorems. The problem is that the intrinsic Besov space norms on the submanifold C measure only the increments between points both on C. However, existing trace theorems imply that every element $f \in B^{\sigma_p'}_{p,\infty}(C)$, $\sigma_p' = \sigma_p - (d-1)/p$ is the restriction to C of some element $\tilde{f} \in B^{\sigma_p}_{p,\infty}(\Lambda)$. The result we need follows if the semi-norm $\|f\|_{\sigma_p'} = \sup_{|\mathbf{r}|\leq\rho} \frac{1}{|\mathbf{r}|^{\sigma_p'}}\left[\int_C ds|\delta\tilde{f}(\mathbf{r})|^p\right]^{1/p}$ is equivalent to the standard Besov semi-norm on $B^{\sigma_p'}_{p,\infty}(C)$. Here $\mathbf{r}$ ranges over a ball of radius ρ inside Λ

random advection—has shown that Lagrangian particle trajectories $\mathbf{x}(t)$, $\mathbf{x}'(t)$ can explosively separate even when $\mathbf{x}_0 = \mathbf{x}'_0$ initially, if the advecting velocity field is only Hölder continuous and not Lipschitz. See Bernard et al. (1998). Mathematically, this is a consequence of the non-uniqueness of solutions to the initial-value problem, while, physically, it corresponds to the two-particle turbulent diffusion of Richardson (1926). Le Jan & Raimond (2002, 2004) have rigorously proved that there is a random process of Lagrangian particle paths $\mathbf{x}(t)$ in the Kraichnan model for a fixed realization of the advecting velocity and a fixed initial particle position. This phenomenon has been termed *spontaneous stochasticity* (Chaves et al. (2003)). A similar notion of "generalized flow" was proposed by Brenier (1989) for the problem of minimizing the action (4). In his formulation, the action is generalized to a functional $S[P] = \frac{1}{2}\int P(d\mathbf{x}) \int_{t_0}^{t_f} dt\, |\dot{\mathbf{x}}(t)|^2$, where P is a probability measure on path-space, and he showed that minimizers always exist in this framework. Unfortunately, this notion does not permit one to define the concept of material lines and surfaces for ideal flow. A more natural generalization of the classical action would be of the form

$$S[P] = \frac{1}{2}\int P(d\mathbf{x}) \int_{t_0}^{t_f} dt \int_{\Lambda} d\mathbf{a}\; |\dot{\mathbf{x}}(\mathbf{a},t)|^2 \tag{10}$$

where P is now a probability measure on time-histories of measure-preserving maps [3]. For any realization of such a random process and for any initial curve C the advected object $C(t) = \mathbf{x}(C,t)$ is well-defined and remains a (random)

[3] The Kraichnan model might also benefit from a formulation in terms of maps. Formally, a group of Markov transition operators $S^{\mathbf{u}}_{t,t'}$ can be defined on spaces of functionals of maps, with a fixed realization of the velocity $\mathbf{u}$, via a Krylov-Veretennikov expansion:

$$S^{\mathbf{u}}_{t,t'} = \sum_{n=0}^{\infty}(-1)^n \int_{t'}^{t} dt_1 \int_{\Lambda} d\mathbf{a}_1 \int_{t'}^{t_1} dt_2 \int_{\Lambda} d\mathbf{a}_2 \cdots \int_{t'}^{t_{n-1}} dt_n \int_{\Lambda} d\mathbf{a}_n$$
$$e^{(t-t_1)\mathcal{L}_0}\left(\mathbf{u}(\mathbf{x}(\mathbf{a}_1),t_1)\cdot\frac{\delta}{\delta\mathbf{x}(\mathbf{a}_1)}\right) e^{(t_1-t_2)\mathcal{L}_0}\left(\mathbf{u}(\mathbf{x}(\mathbf{a}_2),t_2)\cdot\frac{\delta}{\delta\mathbf{x}(\mathbf{a}_2)}\right) e^{(t_2-t_3)\mathcal{L}_0}$$
$$\cdots e^{(t_{n-1}-t_n)\mathcal{L}_0}\left(\mathbf{u}(\mathbf{x}(\mathbf{a}_n),t_n)\cdot\frac{\delta}{\delta\mathbf{x}(\mathbf{a}_n)}\right) e^{(t_n-t')\mathcal{L}_0}.$$

Cf. Le Jan & Raimond (2002, 2004). Here the time-integrals should be defined in the Ito sense with respect to the white-noise velocity field $\mathbf{u}(\mathbf{x},t)$ and $\mathcal{L}_0$ is formally the infinitesimal generator of a diffusion process on the space of maps, given by $\mathcal{L}_0 = \frac{1}{2}\int_\Lambda d\mathbf{a} \int_\Lambda d\mathbf{a}' D_{ij}(\mathbf{x}(\mathbf{a}) - \mathbf{x}(\mathbf{a}'))\frac{\delta^2}{\delta x_i(\mathbf{a})\delta x_j(\mathbf{a}')}$. The Gaussian random velocity has covariance $\langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t')\rangle = D_{ij}(\mathbf{x}-\mathbf{x}')\delta(t-t')$. It would be very interesting to give rigorous meaning to this expansion, especially for the case where the advecting velocity is only Hölder continuous but not Lipschitz in space curve for all time t, if the maps are continuous in space.

Let us assume for the moment that the (very nontrivial) problem can be solved to construct such a generalized flow $\mathbf{x}(\mathbf{a},t)$, or stochastic process in the space of volume-preserving maps, which is hopefully a.s. Hölder continuous in space so that material loops $C(t)$ exist as random, fractal curves. We would like to present some plausibility arguments in favor of the conjecture that circulations shall be conserved in a statistical sense. More precisely, we expect that the circulations $\Gamma(C,t)$ for any initial smooth loop C shall be martingales of the generalized flow:

$$E\left[\Gamma(C,t)|\Gamma(C,\tau),\tau<t'\right]=\Gamma(C,t'),\ \text{ for } t>t'. \tag{11}$$

Here $E[\cdot]$ denotes the expectation over the ensemble of random Lagrangian paths and we have conditioned on the past circulation history $\{\Gamma(C,\tau),\tau<t'\}$. Heuristically,

$$(d/dt)E\left[\Gamma(C,t)|\Gamma(C,\tau),\tau<t'\right]=\lim_{\ell\to 0}E\left[K_\ell(C,t)|\Gamma(C,\tau),\tau<t'\right]. \tag{12}$$

Note that the conditioning event involves scales of the order of the radius of gyration of the loops $C(\tau),\ \tau<t'$, while the circulation-flux involves velocity-increments over separation lengths $\ell\to 0$. Therefore, we expect that Kolmogorov's idea of small-scale homogeneity (and isotropy) will apply. Note, however, that the homogeneous average of the subgrid force $\mathbf{f}_\ell$ is zero, because it is the divergence of the stress tensor. From another point of view, the subgrid force will become increasingly irregular for $\ell\ll R(t)$ (the radius of the loop $C(t)$) and the sign of the integrand $\mathbf{f}_\ell(\overline{\mathbf{x}}_\ell(s,t))\cdot\overline{\mathbf{x}}'_\ell(s,t)$ will oscillate more rapidly as a function of the arclength s. Thus, cancellations will occur. For these reasons, we expect that the limit on the RHS of (12) shall vanish, implying (11). Another formal argument can be given by applying the Noether theorem to the generalized action (10) and using the fact that a global minimizer must also minimize the action for the time segment $[t',t_f]$. On the other hand, based upon our earlier theorem, it is not likely that circulation-flux will vanish as $\ell\to 0$ in every realization, without any averaging.

In this section we have clearly indulged in some speculative thinking, but we hopefully have also succeeded in outlining the various difficulties in properly formulating Kelvin's theorem for turbulent solutions of the Euler equations. Our own view is that the Taylor (1938) mechanism of vortex line-stretching is the underlying cause of enhanced dissipation in three-dimensional turbulence asymptotically at high Reynolds numbers. However, much work remains to elucidate the details of the subtle dynamics involved.

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